Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods
MATH 6720 – Section 001 – Spring 2023
Homework 2
Complex integration

Due: Friday, Feb 17, 2023

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises:	2.4.1
	2.4.4
	2.4.8
	2.5.2
	2.5.3

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2.4.1. From the basic definition of complex integration, evaluate the integral $\oint_C f(z)\,\mathrm{d}z$, where C is the parameterized unit circle enclosing the origin, $C:\ x(t) = \cos t,\ y(t) = \sin t$ or $z = e^{it}$, and where $f(z)$ is given by,

(a) z^2
(b) $\overline{z}^2$
(c) $\frac{z+1}{z^2}$

Solution:

(a) We parameterize the unit circle with $0 \leq t \leq 2\pi$, and since $z = e^{it}$ use,

$$\mathrm{d}z = ie^{it}\,\mathrm{d}t,$$

to write the integral:

$$\int_C z^2\,\mathrm{d}z = \int_0^{2\pi} \left(e^{it}\right)^2 ie^{it}\,\mathrm{d}t = \int_0^{2\pi} ie^{3it}\,\mathrm{d}t = 0$$

(b) With the same parameterization as the previous part, we have,

$$\int_C \overline{z}^2\,\mathrm{d}z = \int_0^{2\pi} \left(e^{-it}\right)^2 ie^{it}\,\mathrm{d}t = i\int_0^{2\pi} e^{-it}\,\mathrm{d}t = 0$$

(c) With the same parameterization as the previous part, we have,

$$\int_C \frac{z+1}{z^2}\,\mathrm{d}z = \int_0^{2\pi} \frac{e^{it}+1}{e^{2it}} ie^{it}\,\mathrm{d}t = i\int_0^{2\pi}(1+e^{-it})\,\mathrm{d}t = 2\pi i.$$

2.4.4. Use the principal branch of $\log z$ and $z^{1/2}$ to evaluate,

(a) $\int_{-1}^{1} \log z\,\mathrm{d}z$

(b) $\int_{-1}^{1} z^{1/2}\,\mathrm{d}z$

Solution:

(a) These integrals can be recast as real-valued integrals. To begin, we recall that for a real variable x:

$$\int_0^1 \log x\,\mathrm{d}x = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \log x\,\mathrm{d}x \overset{\text{IbP}}{=} \lim_{\epsilon\downarrow 0}\left(y\log y - y\right)\Big|_\epsilon^1 = \lim_{\epsilon\downarrow 0}\left(-1-\epsilon\log\epsilon+\epsilon\right) = -1.$$

We now write,

$$\int_{-1}^1 \log z\,\mathrm{d}z = \int_{-1}^0 \log z\,\mathrm{d}z + \int_0^1 \log z\,\mathrm{d}z.$$

The second integral, being an integral of a real-valued function over a real interval, takes value -1 as we have already established. Since we are on the principal branch of the logarithm, then $\log z = \log|z| + i\arg z$, where $\arg z \in [-\pi,\pi)$. In our case, $z = |z|e^{-i\pi}$ for $|z|\in[0,1]$, so we have:

$$\int_{-1}^0 \log z\,\mathrm{d}z = \int_{-1}^0 \left(\log|z| - i\pi\right)\,\mathrm{d}z = \int_0^1 \left(\log x - i\pi\right)\,\mathrm{d}x = -i\pi - 1.$$

Putting everything together, we have:

$$\int_{-1}^1 \log z\,\mathrm{d}z = \int_{-1}^0 \log z\,\mathrm{d}z + \int_0^1 \log z\,\mathrm{d}z = -i\pi - 1 - 1 = -2 - i\pi.$$

(b) For the principal branch of the square root function, we treat $z = |z|e^{i\arg z}$, with $\arg z \in [-\pi,\pi)$. I.e., the integral we wish to compute takes the form,

$$\begin{aligned}\int_{-1}^1 z^{1/2}\,\mathrm{d}z &= \int_{-1}^0 z^{1/2}\,\mathrm{d}z + \int_0^1 z^{1/2}\,\mathrm{d}z \\ &= e^{-i\pi/2}\int_{-1}^0 |z|^{1/2}\,\mathrm{d}z + \int_0^1 z^{1/2}\,\mathrm{d}z = (1-i)\int_0^1 z^{1/2}\,\mathrm{d}z,\end{aligned}$$

where we have used the fact that the value of $|z|^{1/2}$ on $[-1,0]$ equals (a reflection of) that of $z^{1/2}$ on $[0,1]$. This last integral is directly computable via the parameterization:

$$z(t) = t, \qquad\qquad t\in[0,1],$$

i.e.,

$$\int_0^1 z^{1/2}\,\mathrm{d}z = \int_0^1 \sqrt{t}\,\mathrm{d}t = \frac{2}{3},$$

and hence,

$$\int_{-1}^1 z^{1/2}\,\mathrm{d}z = (1-i)\frac{2}{3}.$$

2.4.8. Let C be an arc of the circle $|z| = R$, $(R > 1)$ of angle $\pi/3$. Show that

$$\left| \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \right| \leq \frac{\pi}{3}\left(\frac{R}{R^3-1}\right), \tag{1}$$

and deduce $\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z = 0$.

Solution: We seek to use the result stating that if $|f(z)| \leq M$ over a contour C of arclength L, then,

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq ML. \tag{2}$$

The length of this contour (a radius-R circular arc of angle $\pi/3$) has value $L = R\pi/3$. To compute M for $f(z) = \frac{1}{z^3+1}$, we note that

$$\max_{z\in C} |f(z)| \leq \max_{|z|=R} |f(z)| = \max_{|z|=R} \left|\frac{1}{z^3+1}\right| = \frac{1}{\min_{|z|=R} |z^3+1|}.$$

We compute the desired minimum via the (reverse) triangle inequality:

$$\min_{|z|=R} |z^3+1| \geq \min_{|z|=R} |z^3| - 1 = R^3 - 1.$$

Hence, we have

$$\max_{z\in C} |f(z)| \leq \frac{1}{R^3-1} \coloneqq M.$$

Using $L = R\pi/3$ and $M = 1/(R^3-1)$ in (2) proves (1). The subsequent limit is immediate:

$$\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \leq \lim_{R\to\infty} \left| \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \right| \leq \lim_{R\to\infty} \frac{\pi}{3}\frac{R}{R^3-1} = 0.$$

2.5.2. Use partial fractions to evaluate the following integrals $\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

(a) $\frac{1}{z(z-2)}$
(b) $\frac{z}{z^2-1/9}$
(c) $\frac{1}{z(z+\frac{1}{2})(z-2)}$

Solution:

(a) We know that for an arbitrary complex number a,

$$\oint_C \frac{1}{z-a}\,\mathrm{d}z = \begin{cases} 0, & a \text{ is outside } C \\ 2\pi i, & a \text{ is inside } C \end{cases} \tag{3}$$

We will use this property to evaluate the integral once we have expanded in partial fractions. $f(z)$ has poles at $z = 0, z = 2$, so we use the ansatz,

$$f(z) = \frac{C_1}{z} + \frac{C_2}{z-2}.$$

By clearing denominators, this leads to the following linear system for the unknowns C_1, C_2:

$$\left.\begin{array}{c} -2C_1 = 1, \\ C_1 + C_2 = 0 \end{array}\right\} \Longrightarrow (C_1, C_2) = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Hence,

$$\oint_C f(z)\,\mathrm{d}z = -\frac{1}{2}\oint_C \frac{1}{z}\,\mathrm{d}z + \frac{1}{2}\oint_C \frac{1}{z-2}\,\mathrm{d}z \overset{(3)}{=} -\pi i$$

(b) The partial fractions ansatz in this case is,

$$f(z) = \frac{C_1}{z - 1/3} + \frac{C_2}{z + 1/3},$$

resulting in the linear system,

$$\left.\begin{array}{c} C_1 + C_2 = 1, \\ C_1 - C_2 = 0 \end{array}\right\} \Longrightarrow (C_1, C_2) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Therefore,

$$\oint_C f(z)\,\mathrm{d}z = \frac{1}{2}\oint_C \frac{1}{z-1/3}\,\mathrm{d}z + \frac{1}{2}\oint_C \frac{1}{z+1/3}\,\mathrm{d}z \overset{(3)}{=} \pi i + \pi i = 2\pi i$$

(c) The partial fractions ansatz for this function is,

$$f(z) = \frac{C_1}{z} + \frac{C_2}{z+\frac{1}{2}} + \frac{C_3}{z-2},$$

resulting in the linear system,

$$\left.\begin{array}{c} C_1 + C_2 + C_2 = 0 \\ -\frac{3}{2}C_1 - 2C_2 + \frac{1}{2}C_3 = 0 \\ -C_1 = 1, \end{array}\right\} \Longrightarrow (C_1, C_2, C_3) = \left(-1, \frac{4}{5}, \frac{1}{5}\right)$$

Therefore,

$$\oint_C f(z)\,\mathrm{d}z = -1\oint_C \frac{1}{z}\,\mathrm{d}z + \frac{4}{5}\oint_C \frac{1}{z+1/2}\,\mathrm{d}z + \frac{1}{5}\oint_C \frac{1}{z-2}\,\mathrm{d}z \overset{(3)}{=} -2\pi i + \frac{8}{5}\pi i = -\frac{2}{5}\pi i.$$

2.5.3. Evaluate the following integral,

$$\oint_C \frac{e^{iz}}{z(z-\pi)}\,\mathrm{d}z,$$

for each of the following four cases (all circle are centered at the origin; use Eq. (1.2.19) as necessary).

(a) C is the boundary of the annulus between circles of radius 1 and radius 3.
(b) C is the boundary of the annulus between circles of radius 1 and radius 4.
(c) C is a circle of radius R, where $R > \pi$.
(d) C is a circle of radius R, where $R < \pi$.

Solution: Before beginning the exercises, we make the following computation, which will be useful:

$$
\begin{aligned}
f(z) := \frac{e^{iz}}{z(z-\pi)} &\overset{\text{partial fractions}}{=} -\frac{1}{\pi}\frac{e^{iz}}{z} + \frac{1}{\pi}\frac{e^{iz}}{z-\pi} \\
&= -\frac{1}{\pi}\frac{e^{iz}}{z} + \frac{e^{i\pi}}{\pi}\frac{e^{i(z-\pi)}}{z-\pi} \\
&\overset{\text{Eqn. (1.2.19)}}{=} -\frac{1}{\pi}\frac{\sum_{k=0}^{\infty}\frac{z^k}{k!}}{z} - \frac{1}{\pi}\frac{\sum_{k=0}^{\infty}\frac{(z-\pi)^k}{k!}}{(z-\pi)} \\
&= -\frac{1}{\pi z} - \frac{1}{\pi(z-\pi)} + \underbrace{\frac{1}{\pi}\left(-\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!} - \sum_{k=1}^{\infty}\frac{(z-\pi)^{k-1}}{k!}\right)}_{g(z)} \\
&= -\frac{1}{\pi z} - \frac{1}{\pi(z-\pi)} + g(z),
\end{aligned}
$$

where $g(z)$ is entire. Since $g(z)$ is entire, then for a curve C enclosing a connected or multiply connected region, we have,

$$
\oint_C f(z)\,\mathrm{d}z = -\frac{1}{\pi}\oint_C \left(\frac{1}{z} + \frac{1}{z-\pi}\right)\mathrm{d}z + \oint_C g(z)\,\mathrm{d}z = -\frac{1}{\pi}\oint_C \left(\frac{1}{z} + \frac{1}{z-\pi}\right)\mathrm{d}z. \tag{4}
$$

We will use the above property to compute solutions to this problem:

(a) This curve C does not enclose the points $z = 0$ or $z = \pi$, and hence h is analytic inside the enclosed region, so that by the Cauchy-Goursat Theorem,

$$
\oint_C f(z)\,\mathrm{d}z = 0.
$$

(b) This curve C encloses the point $z = \pi$, but not $z = 0$. Hence,

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\
&\overset{(3)}{=} -\frac{1}{\pi}(0 + 2\pi i) = -2i.
\end{aligned}
$$

(c) This region includes both points $z = 0$ and $z = \pi$. Therefore,

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\
&\overset{(3)}{=} -\frac{1}{\pi}(2\pi i + 2\pi i) = -4i.
\end{aligned}
$$

(d) This region includes $z = 0$, but not $z = \pi$. Therefore,

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\
&\overset{(3)}{=} -\frac{1}{\pi}(2\pi i + 0) = -2i.
\end{aligned}
$$