Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2023

Homework 0 Solutions

Basics of complex numbers

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 1.1.2 (b,d)
1.1.3
1.2.1
1.2.8
1.3.1
1.3.5

1.1.2. Express each of the following in the form $a + bi$, where a and b are real:
(b) $\frac{1}{1+i}$
(d) $|3 + 4i|$

Solution:
(b) We multiply both numerator and denominator by the complex conjugate:

$$\frac{1}{1+i} = \frac{(1-i)}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - i\frac{1}{2},$$

i.e., $a = 1/2$ and $b = -1/2$.
(d) This is a purely real number:

$$|3 + 4i| = \sqrt{3^2 + 4^2} = 5 = 5 + 0i,$$

so that $a = 5$, $b = 0$.

1.1.3. Solve for the roots of the following equations:
(a) $z^3 = 4$
(b) $z^4 = -1$
(c) $(az + b)^3 = c$, where $a, b, c > 0$
(d) $z^4 + 2z^2 + 2 = 0$

Solution:
(a) We write

$$4 = 4e^{i0} \quad \Longrightarrow \quad z = \sqrt[3]{4}e^{i2\pi/3},\ \sqrt[3]{4}e^{i2\pi/3},\ \sqrt[3]{4}e^{i4\pi/3}$$

(b) We have,

$$-1 = 1e^{i\pi} \quad \Longrightarrow \quad z = e^{i\pi/4},\ e^{i3\pi/4},\ e^{5\pi/4},\ e^{7\pi/4}$$

(c) Letting $w = az + b$, then we seek the roots w such that $w^3 = c$ with $c > 0$. Therefore,

$$w = \sqrt[3]{c},\ \sqrt[3]{c}e^{i2\pi/3},\ \sqrt[3]{c}e^{i4\pi/3}.$$

Therefore, $z = (w - b)/a$ takes values,

$$z = \frac{1}{a}(\sqrt[3]{c} - b),\ \frac{1}{a}(\sqrt[3]{c}e^{i2\pi/3} - b),\ \frac{1}{a}(\sqrt[3]{c}e^{i4\pi/3} - b),$$

(d) We have,

$$z^4 + 2z^2 + 2 = 0 \quad \Longrightarrow \quad (z^2 + 1)^2 + 1 = 0,$$

and hence $z^2 + 1$ are the second roots of -1,

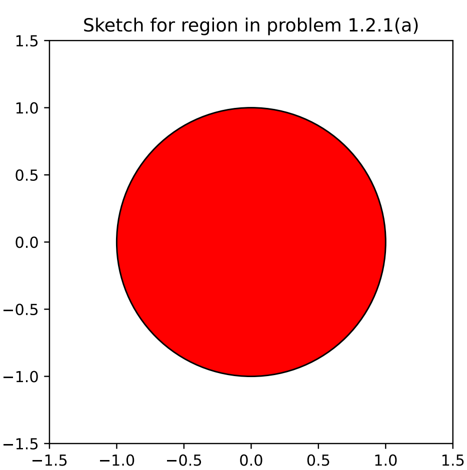
$$z^2 + 1 = e^{i\pi/2},\ e^{i3\pi/2} \quad \Longrightarrow \quad z^2 = -1 \pm i = \sqrt{2}e^{3\pi/4},\ \sqrt{2}e^{5\pi/4}.$$

Thus, z takes on the 4 values,

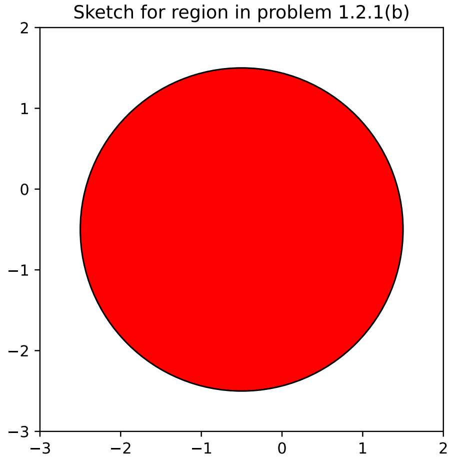
$$z = \sqrt[4]{2}e^{3\pi/8},\ \sqrt[4]{2}e^{11\pi/8}, \qquad z = \sqrt[4]{2}e^{5\pi/8},\ \sqrt[4]{2}e^{13\pi/8}.$$

1.2.1. Sketch the regions associated with the following inequalities. Determine if the region is open, closed, bounded, or compact.
(a) $|z| \leq 1$
(b) $|2z + 1 + i| < 4$
(c) $\mathrm{Re}\,(z) \geq 4$
(d) $|z| \leq |z + 1|$
(e) $0 < |2z - 1| \leq 2$

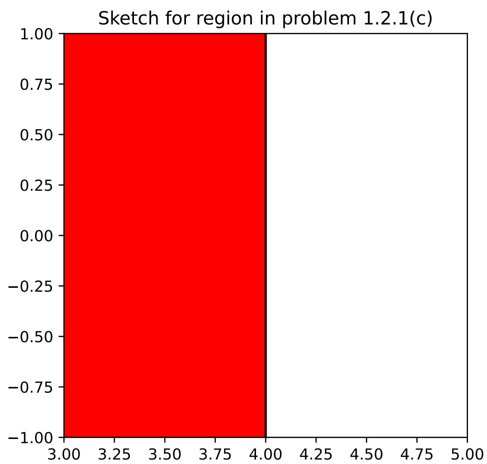
Solution:
(a) The region is closed, bounded, and compact.



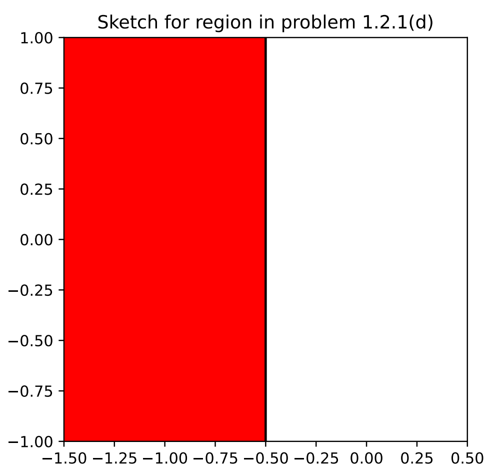
(b) The region is open and bounded.



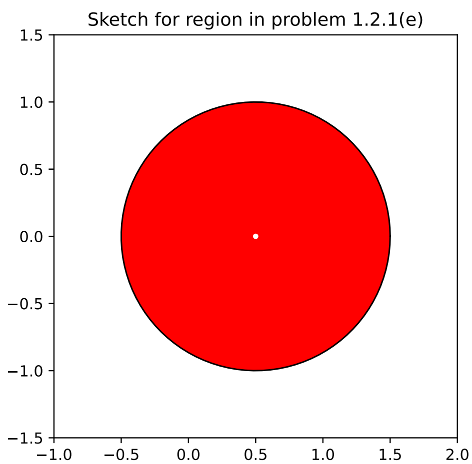
(c) The region is closed.



(d) The region is closed.



(e) The region is bounded.



1.2.5. Use any method to determine series expansions for the following functions:
(a) $\frac{\sin z}{z}$
(b) $\frac{\cosh z - 1}{z^2}$
(c) $\frac{e^z - 1 - z}{z}$

Solution: Using formulas for known power series, we have:
(a)

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!}$$

(b)

$$\frac{\cosh z - 1}{z^2} = \frac{1}{z^2} \left(-1 + \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \right) = \frac{1}{z^2} \sum_{j=1}^{\infty} \frac{z^{2j}}{(2j)!} = \sum_{j=1}^{\infty} \frac{z^{2j-2}}{(2j)!}$$

(c)

$$\frac{e^z - 1 - z}{z} = \frac{1}{z} \left(-1 - z + \sum_{j=0}^{\infty} \frac{z^j}{j!} \right) = \frac{1}{z} \sum_{j=2}^{\infty} \frac{z^j}{j!} = \sum_{j=2}^{\infty} \frac{z^{j-1}}{j!} = \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j+2)!}$$

1.2.6. Let $z_1 = x_1$ and $z_2 = x_2$ with x_1, x_2 real, and the relationship,

$$e^{i(x_1+x_2)} = e^{ix_1} e^{ix_2},$$

to deduce the known trigonometric formulae,

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2 \tag{1a}$$
$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2, \tag{1b}$$

and therefore show,

$$\sin 2x = 2 \sin x \cos x \tag{2a}$$
$$\cos 2x = \cos^2 x - \sin^2 x \tag{2b}$$

Solution: Using Euler's identity, we have,

$$\begin{aligned}\cos(x_1+x_2)+i\sin(x_1+x_2) = e^{i(x_1+x_2)} = e^{ix_1}e^{ix_2} &= (\cos x_1 + i\sin x_1)(\cos x_2 + i\sin x_2)\\ &= (\cos x_1\cos x_2 - \sin x_1 \sin x_2) + i\,(\sin x_1 \cos x_2 + \cos x_1 \sin x_2\end{aligned}$$

The real and imaginary parts of the left- and right-hand sides above must be equal, implying (1). The relation (2) is a directly result of (1) by setting $x_1 = x_2 = x$.

1.2.8. Consider the transformation,

$$w = z + 1/z, \qquad z = x + iy, \qquad w = u + iv.$$

Show that the image of the points in the upper half z plane ($y > 0$) that are exterior to the circle $|z| = 1$ corresponds to the entire upper half plan $v > 0$.

Solution: Note that,

$$\begin{aligned}u &= \mathrm{Re}\,(w) = \mathrm{Re}\,(z+1/z) = x + \mathrm{Re}\,(1/z) = x + \mathrm{Re}\left(\frac{\overline{z}}{|z|^2}\right) = x\left(1 + \frac{1}{|z|^2}\right),\\ v &= \mathrm{Im}\,(w) = \mathrm{Im}\,(z+1/z) = y + \mathrm{Im}\,(1/z) = y + \mathrm{Im}\left(\frac{\overline{z}}{|z|^2}\right) = y\left(1 - \frac{1}{|z|^2}\right).\end{aligned}$$

Thus, if both $y > 0$ and z is exterior to $|z| = 1$, i.e., if $y > 0$ and $|z| > 1$, then

$$v = y\left(1 - \frac{1}{|z|^2}\right) > 0,$$

so that $z \mapsto w$ for $|z| > 1$ and $y > 0$ maps onto the upper half plane $v > 0$. To show that the image of this map is exactly the upper half plane, set $r = |z|$, and note that from the first expressions,

$$\left(\frac{ru}{r^2+1}\right)^2 + \left(\frac{rv}{r^2-1}\right)^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1,$$

i.e., circles of radius $r > 1$ in the z plane are mapped to ellipses in the w plane. Rewriting the condition above, we have,

$$\left(\frac{u}{R+1}\right)^2 + \left(\frac{v}{R-1}\right)^2 = \frac{1}{R}, \qquad R = r^2.$$

We will use this relation to show that the image of $z \mapsto z + 1/z$ is the open upper half plane. Let $w = u + iv$ be in the upper half plane, i.e., $v > 0$; our overall goal is to construct $z = x + iy$ that maps to w. Define,

$$\begin{aligned}f_1(R) &\coloneqq \left(\frac{u}{R+1}\right)^2 + \left(\frac{v}{R-1}\right)^2,\\ f_2(R) &\coloneqq \frac{1}{R}.\end{aligned}$$

where we now take R as an unknown. We seek to show that there exists some $R > 1$ such that $f_1(R) = f_2(R)$. Note that for R sufficiently large then $f_1(R) \le f_2(R)$ because $f_1 \sim 1/R^2$ and

$f_2 \sim 1/R$. The technical details are:

$$R > 3 + 2\max\{u^2, v^2\} \overset{R>1}{\Longrightarrow} R - 2 + \frac{1}{R} > 2\max\{u^2, v^2\}$$
$$\Longrightarrow \frac{1}{R} > \frac{2\max\{u^2, v^2\}}{(R-1)^2} > \frac{u^2}{(R+1)^2} + \frac{v^2}{(R-1)^2}.$$

This establishes that $f_1(R) \le f_2(R)$ for R sufficiently large. Similarly, for $R > 1$ sufficiently small, we have $f_1(R) \ge f_2(R)$. To establish this, note that since $v > 0$, the inequality,

$$\left(\frac{v}{R-1}\right)^2 \ge \frac{1}{R} \iff R^2 - (v+2)R + 1 < 0,$$

becomes an equality when

$$R_* = 1 + \frac{v}{2} + \frac{1}{2}\sqrt{v^2 + 2v} > 1,$$

and hence for $R \in (1, R_*)$, then

$$\left(\frac{v}{R-1}\right)^2 \ge \frac{1}{R} \Longrightarrow \left(\frac{u}{R+1}\right)^2 + \left(\frac{v}{R-1}\right)^2 \ge \frac{1}{R},$$

establishing that for $R > 1$ sufficiently small, then $f_1(R) \ge f_2(R)$. Since both f_1 and f_2 are continuous functions for $R > 1$, then there must exist some point $R = R(u, v) > 1$ such that

$$f_1(R) = f_2(R) \Longrightarrow \left(\frac{u}{R+1}\right)^2 + \left(\frac{v}{R-1}\right)^2 = \frac{1}{R}.$$

Now set,

$$x = \frac{u}{1 + \frac{1}{R}}, \qquad y = \frac{v}{1 - \frac{1}{R}}.$$

By construction, $z = x + iy$ then satisfies $|z| = \sqrt{x^2 + y^2} = \sqrt{R} > 1$, and $\text{Re}\,(z) = y > 0$ since $v > 0$. This point z maps to (arbitrarily chosen) point $w = u + iv$ in the upper half plane.

1.3.1. Evaluate the following limits,

(a) $\lim_{z\to i}(z + 1/z)$
(b) $\lim_{z\to z_0} 1/z^m$, m integer
(c) $\lim_{z\to i} \sinh z$
(d) $\lim_{z\to 0} \frac{\sin z}{z}$
(e) $\lim_{z\to\infty} \frac{\sin z}{z}$
(f) $\lim_{z\to\infty} \frac{z^2}{(3z+1)^2}$
(g) $\lim_{z\to\infty} \frac{z}{z^2+1}$

Solution:

(a) By direct evaluation, we have,

$$\lim_{z\to i} z + \frac{1}{z} = i - i = 0$$

(b) If $z_0 = 0$, then the limit doesn't exist. If $z_0 \neq 0$, the again by direct evaluation:

$$\lim_{z \to z_0} \frac{1}{z^m} = \frac{1}{z_0^m}$$

(c) By direct evaluation,

$$\lim_{z \to i} \sinh z = \frac{1}{2}(e^i - e^{-i}) = i\mathrm{Im}\left(e^i\right) = i \sin i.$$

(d) Via L'Hôpital's Rule,

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1$$

(e) This limit does not exist. To see why, we compute the numerator for $z = x + iy$:

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left(e^{-y} - e^{y} + 2i \sin x\right).$$

While $|\sin x|$ is bounded by 1 as $|z| \to \infty$, the quantity $e^{-y} - e^{y}$ can approach either 0 ($|z| \to \infty$ with $y = 0$) or $\pm\infty$ ($|z| \to \infty$ with $y \to \pm\infty$). With this behavior, $\sin z/z$ can approach different values as $|z| \to \infty$ and thus the limit does not exist.

(f) This limit exists: we compute it by computing the complementary limit that has equal value,

$$\lim_{z \to 0} \frac{\left(\frac{1}{z}\right)^2}{\left(3\frac{1}{z} + 1\right)^2} = \lim_{z \to 0} \frac{1}{(3+z)^2} = \frac{1}{9}.$$

(g) We again look at the complementary limit,

$$\lim_{z \to 0} \frac{\frac{1}{z}}{\frac{1}{z^2} + 1} = \lim_{z \to 0} \frac{1}{z + \frac{1}{z}} = 0$$

1.3.5. Show that the functions $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ are nowhere differentiable.

Solution: By direct computation, we have,

$$\lim_{w \to 0} \frac{\mathrm{Re}\,(z+w) - \mathrm{Re}\,(z)}{w} = \lim_{w \to 0} \frac{\mathrm{Re}\,(w)}{w}.$$

This limit is not unique; its value depends on how w approaches 0. Thus, $\mathrm{Re}\,(z)$ is nowhere differentiable. A similar computation for $\mathrm{Im}\,(z)$ can be carried out.