Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MTH6720 – Section 01 – Spring 2023

Final exam formula sheet

In what follows, C_R denotes a semicircular arc of radius R in the upper half-plane centered at the origin. The contour C_ϵ is a circular arc of radius ϵ centered around a point z_0 that sweeps out an angle of ϕ.

1. Suppose f is analytic on an open domain containing a simple closed loop C. Then for all integers $n \geq 0$ and all z enclosed by C,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w,$$

2. The coefficients for a Laurent series of the function f are given by,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w$$

3. If a continuous f is bounded over a contour C of finite length, i.e., $|f(z)| \leq M < \infty$ for all $z \in C$ and $\int_C |\,\mathrm{d}z| = L < \infty$, then

$$\left| \int_C f(z) \, \mathrm{d}z \right| \leq ML$$

4. Suppose $f(z) = P(z)/Q(z)$ is a rational function with $\deg Q \geq \deg P + 2$. Then,

$$\lim_{R \to \infty} \int_{C_R} f(z) \, \mathrm{d}z = 0.$$

5. (Jordan's Lemma) Suppose that $f(z) \to 0$ uniformly for $z \in C_R$ as $R \to \infty$. Then for any $k > 0$,

$$\lim_{R \to \infty} \int_{C_R} e^{ikz} f(z) \, \mathrm{d}z = 0.$$

6. Suppose that $(z - z_0) f(z) \to 0$ uniformly for $z \in C_\epsilon$ as $\epsilon \to 0$. Then,

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, \mathrm{d}z = 0.$$

7. Suppose that f has a simple pole at $z = z_0$. Then

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, \mathrm{d}z = i\phi \mathrm{Res}(f; z_0).$$

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8. With C_R any origin-centered circular arc (not necessarily in the upper half-plane), if $zf(z) \to 0$ uniformly on C_R as $R \to 0$, then,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Laplace-type integrals These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{-k\phi(t)}\,\mathrm{d}t$ for $a < b$.

a. (Watson's Lemma) Set $a = 0$ and $\phi(t) = t$. Assume f is integrable with the series expansion,

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad t \to 0^+, \quad \alpha > -1, \quad \beta > 0.$$

In addition, if $b < \infty$ then assume $|f(t)| \le M < \infty$ for $t \in [a, b]$, and if $b = \infty$ then assume $f(t) = \mathcal{O}(e^{ct})$ as $t \to \infty$ for some $c \in \mathbb{R}$. Then,

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{k^{\alpha+\beta n+1}}.$$

b. (Laplace's Method) Assume $b < \infty$, and that $\phi \in C^4([a, b])$ and $f \in C^2([a, b])$. Suppose that for some $c \in [a, b]$, we have $\phi'(c) = 0$ and $\phi''(c) > 0$. Also, assume that $\phi'(t) \neq 0$ for all $t \in [a, b]\backslash\{c\}$. Then,

$$I(k) \sim G(c)e^{-k\phi(c)} f(c)\sqrt{\frac{2\pi}{k\phi''(c)}} + \mathcal{O}\left(\frac{e^{-k\phi(c)}}{k^{G(c)+1/2}}\right), \qquad G(c) := \begin{cases} 1, & c \in (a, b) \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Fourier-type integrals These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{ik\phi(t)}\,\mathrm{d}t$ for $a < b$.

a. Set $a = 0$, and $\phi(t) = \mu t$, where $\mu = \pm 1$, and $k > 0$. Suppose f vanishes infinitely smoothly at $t = b$, that $f \in C^\infty((0, b])$, and that for some $\gamma > -1$, $f(t) \sim t^\gamma + o(t^\gamma)$ as $t \to 0^+$. Then,

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma + 1)e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}).$$

b. (Stationary phase) Suppose $c \in [a, b]$ is the only value of t where $\phi'(t)$ vanishes. Assume that f vanishes infinitely smoothly at both $t = a$ and $t = b$, and that both f and ϕ are C^∞ on the intervals $[a, c)$ and $(c, b]$. Suppose that there is some $\gamma > -1$ such that as $t \to c$,

$$\begin{aligned} \phi(t) - \phi(c) &\sim \alpha(t - c)^2 + o((t - c)^2), \\ f(t) &\sim \beta(t - c)^\gamma + o((t - c)^\gamma). \end{aligned}$$

Then with $\mu = \operatorname{sgn} \alpha$,

$$\int_a^b f(t)e^{ik\phi(t)}\,\mathrm{d}t \sim e^{ik\phi(c)}\beta\Gamma\left(\frac{\gamma + 1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right).$$

Final Exam **Name:** ____________________

MATH6720, Section 01 May 3, 2023

This test is:

- closed-book
- closed-notes (but you may refer to the formula sheet provided)
- no-calculator
- 120 minutes

Indicate your answers clearly, and show your work. Partial credit will be awarded based on work shown. Full credit will not be awarded without some work shown.

There are 5 questions, some with multiple parts. You must complete all questions. Each question is worth a total of 20 points. (100 points total)

The anatomy of this exam is as follows:

- Cover page (this page)
- A single page containing the statements for all 5 questions (one page, no space for work)
- You are given separate blank sheets on which to complete the questions.