Final exam: Friday, Dec. 15
8-10am here
Open notes/book/calculator
Study material: HW problems

Thursday class: no class
office hours: 9:30-10:30, WEB 4666
(also 12-1pm)

Project 2 due Thursday.

Math 5760/6890: Introduction to Mathematical Finance
The Black-Scholes-Merton Model – European Call Options

See Petters and Dong 2016, Sections 8.1-8.2

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The model assumptions

Before discussing details of the Black-Scholes-Merton model, we list some assumptions:

- No-arbitrage
- No transaction costs
- Easy availability of a risk-free security with a(n annual) rate $r > 0$
- Liquidity of assets: fractional shares of any amount are permitted to be bought and sold
- Unlimited short selling permitted
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The main question we'll provide analysis for: For a derivative with the risky asset as underlier, what should the price/premium of the option be?

The notation

We'll use notation that is fairly typical at this point:

- $t = 0$ is today, $t = T > 0$ is a fixed terminal time
- S_t is the (per-unit) underlier price at time t
- $f(S_t, t)$ is the (per-unit of S) price of a derivative with S_t as underlier
  - Typically we know $f(S_T, T)$ (e.g., from a payoff diagram)
  - We want to identify $f(S_0, 0)$, the price at time 0 (the premium)

The hedging portfolio

One basic idea is the following: we will form a portfolio that hedges against the value of the derivative.

I.e., suppose we hold one share of the derivative with price f – we seek to create a portfolio that hedges against the value of the derivative as it fluctuates with the underlier price.

So what is the change in f with respect to changes in S?

$f = f(s, t)$ $\qquad \frac{\partial f}{\partial s}$?

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Hence, we can hedge against f by purchasing shares in S:

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The idea here: The change in value of f can be offset by holding $\frac{\partial f}{\partial S}$ shares of S.

Therefore, let's create a portfolio P that shorts one unit of the option, and an appropriate number of shares of S to hedge:

$$\mathrm{d}P = -\mathrm{d}f + \frac{\partial f}{\partial S}\mathrm{d}S$$

i.e.,

$$P = -f + \frac{\partial f}{\partial S}S.$$

Delta hedging

The portfolio construction strategy we've just described is called (instantaneous) delta-hedging.

- We hedge according to the "delta", $\frac{\partial f}{\partial S}$, of the derivative.
- This requires instantaneous buying/selling of S.

$$\mathrm{d}P = -\mathrm{d}f + \frac{\partial f}{\partial S}\mathrm{d}S.$$

Itô processes

$$dP = -df + \frac{\partial f}{\partial S} dS.$$

$f = f(S_t, t)$

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Part of the Black-Scholes-Merton modeling assumption is that the underlier evolves according to a geometric Brownian motion:

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}B, \qquad\qquad S(0) = S_0,$$

where B is a standard Brownian motion, and (μ, σ) are the continuous-time drift and volatility, respectively.

$f = f(S_t, t)$ $\qquad$ $dS = \mu dt + \sigma dB_t$

$$df \sim dt\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right]$$

$$+ dB_t\left[\frac{\partial f}{\partial S}\sigma\right]$$

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Now recall Itô's Lemma: a function of an Itô process is another Itô process, and its corresponding SDE can be written as functions of the original SDE.

Applying this to $f(S_t, t)$:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t + \sigma S \frac{\partial f}{\partial S}\mathrm{d}B$$

The delta-hedge portfolio

Putting this all together, we have the following evolution law for the delta-hedge portfolio:

$$\mathrm{d}P = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t,$$

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In a no-arbitrage market, the only way this portfolio is an efficient one is if it evolves according to the risk-free security:

$$\mathrm{d}P = rP\mathrm{d}t. \quad \text{— } P(t) \sim e^{rt}$$

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Then recalling our formula for P

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t = \mathrm{d}P = rP\mathrm{d}t = r\left(-f + \frac{\partial f}{\partial S} S\right)\mathrm{d}t,$$

i.e.,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

The Black-Scholes equation

The PDE we have just derived is called the Black-Scholes (partial differential) equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

It is typically supplemented with boundary and *terminal* conditions:

$$\begin{aligned} f(S,T) &= \text{payoff function} \\ f(0,t) &= 0 \text{ for all time} \\ f(x,t)&\text{for large } x \end{aligned}$$

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The goal is to identify/compute a solution to the Black-Scholes equation, i.e., $f(s,0)$.

For sufficiently complicated examples (e.g., non-constant μ, σ), this equation is numerically solved.

However, in simplified cases, we can compute exact solutions.

European options

For a European call option, the payoff is,

$$f(s,T) = \max\{s-K, 0\}.$$

Our asymptotic condition is,

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We can solve this equation analytically (though we'll omit most steps). The basic ideas:

- Reverse time: $\tau = T - t$.
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These transformations make the PDE a rather familiar one:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}, \qquad\qquad u(x,0) = K(e^x - 1)H(x),$$

with $H(x)$ the Heaviside function.

This can be solved with somewhat standard methods, e.g., using the heat kernel:

$$u(x,\tau) = \int_{-\infty}^{\infty} u_0(y) G(x,y,\tau) \mathrm{d}y, \qquad G(x,y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(\frac{-(x-y)^2}{2\sigma^2\tau}\right).$$

Once $u(x,T)$ is computed, we have $f(s,0)$.

European call

The solution for the European call is:

$$f(s,t) = \Phi(d_+)s - \Phi(d_-)Ke^{-r(T-t)},$$

← Black-Scholes formula

where $\Phi(\cdot)$ is the CDF of the standard normal:

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x,$$

and $d_\pm$ are given by,

$$\begin{aligned} d_+ &= \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right), \\ d_- &= d_+ - \sigma\sqrt{T-t} \end{aligned}$$

Note that this allows us to price the derivative for any $t \in [0, T]$.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.