Math 5760/6890: Introduction to Mathematical Finance
Forwards and options

See Petters and Dong 2016, Sections 7.2-7.5

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

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Our current status

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(Typically assuming statistics like drift and volatility.)

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Forward modeling of asset prices is a good outcome, but one of the real practical payoffs for this type of stochastic model is that of being able to price *derivatives*, i.e., interparty contracts whose value is derived from the underlying assets.

We'll spend some time reviewing basic derivatives before we come to a fundamental technical strategy for pricing them: the Black-Scholes-Merton model.

The basic building block: forward contracts

Forward contracts are simple. They are agreements

- typically between two parties, the *buyer* and *seller*
- that specify a transaction to happen at a specified time and quantity
  - the buyer agrees to purchase an *underlier*, i.e., the asset, from the seller
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  - the *delivery date* or *expiry* is the future transaction date
- with a specified *forward price* and *contract size* of the underlier
- in which typically no payment is made upon initiation.
- that are *not* standardized.
  - payment details can be arbitrary
  - e.g., a down payment, involving payment of a portion of the forward price, may be involved

Forward contracts in practice

Forward contracts are simple enough to consider, and are quite popular:

- businesses often engage in vanilla forward contracts to ensure timely and predictable availability of products to customers
- individuals often enter into subscription-type forward contracts for commodities
- speculators will enter into forward contracts as a profit-making strategy

The temporal value of forward contracts

The *value* of a forward contract depends on time.
That is, the underlier can change value despite an initially agreed upon forward price.

- Let $t = 0$ be the time at which a forward contract is initiated
- Let $t = T$ be the delivery date
- Let $F_T(0)$ be the forward price: the per-unit price agreed upon at time 0 that comes due at time T.
- Let $S(t)$ denote the per-unit underlier price
- The buyer's per-unit "profit", or the seller's loss, is $S(T) - F_T(0)$.

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If we model uncertainty about the future as randomness in S, then $S(T) - F_T(0)$ is random.

What's more: this randomness depends on the current value of t.

I.e., conditioned on the current time t, the random processes,

$$S(T|t = 0), \qquad S(T|t = t > 0),$$

have different distributions. Hence, the forward contract's long position payoff,

$$S(T|t) - F_T(0),$$

is a t-dependent random process.

Options

Options are more complicated versions of standardized forward contracts.

Recall:

- The buyer pays a premium at the initiation of the option contract
- The buyer purchases the right to conduct a transaction at a specified *strike* price in the future. (This is the options analog of the "forward price".)
  - (European-style) In a simple case, this option may be exercised only at expiry, but not before.
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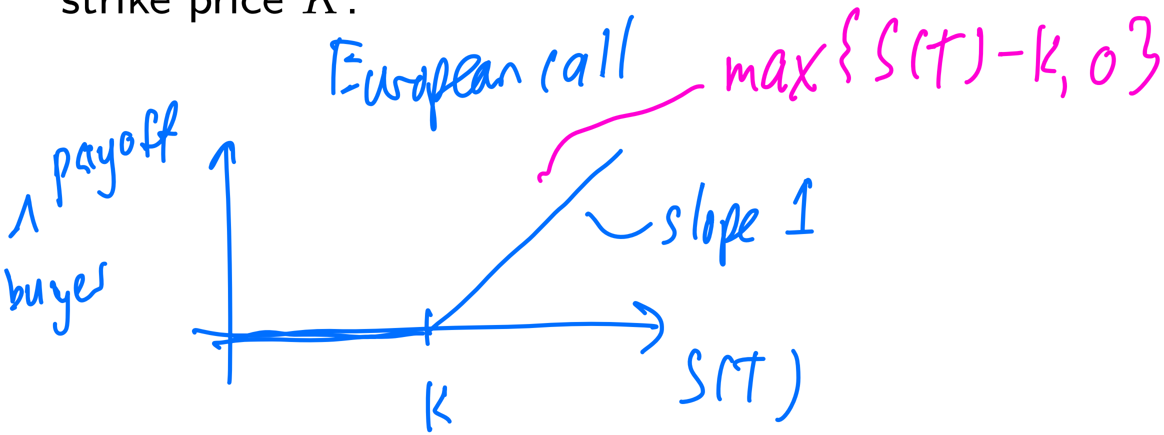
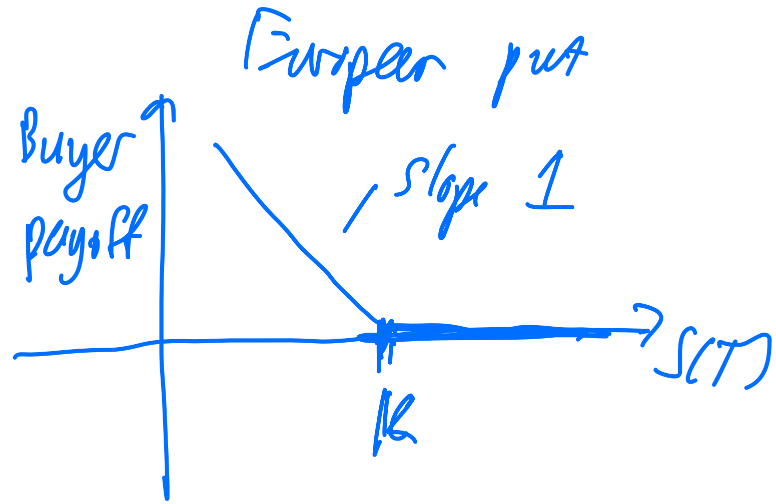
Note again, that the payoff $\max\{S(T) - K, 0\}$ is random and depends on the current time t!

Profit diagrams

Visualizing options payoffs can be more intuitive:

Recall that, for European options, a payoff diagram plots the per-unit payoff versus the spot price $S(T)$. E.g., we can draw payoff diagrams for European call and put options with strike price K.





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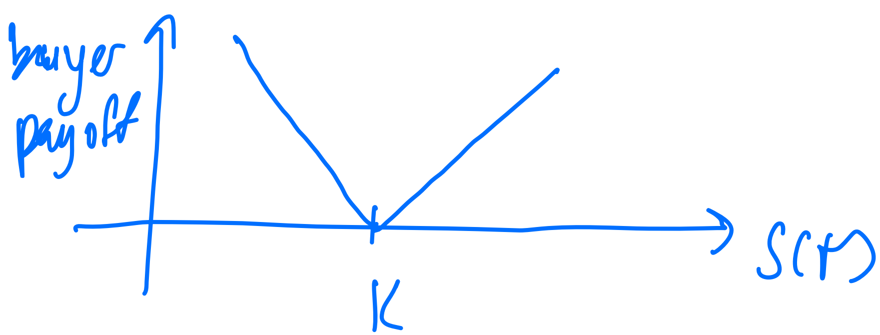
Portfolios can be constructed that hold *both* long call and put options.

E.g., what is the payoff diagram for a portfolio that holds simultaneuosly a long European call and put option for a single share of an asset with strike price K?

$$\max\{S(T)-K, 0\} + \max\{K-S(T), 0\}$$



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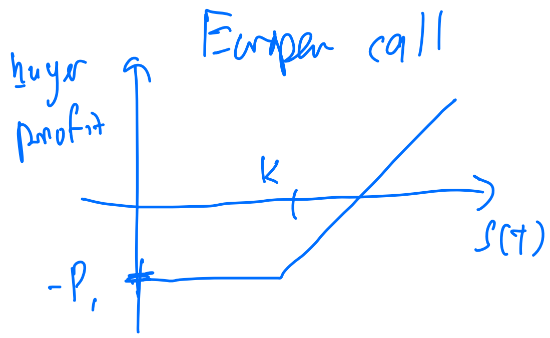
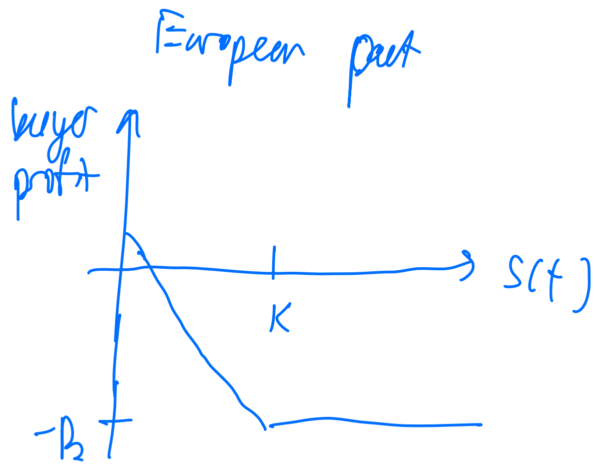
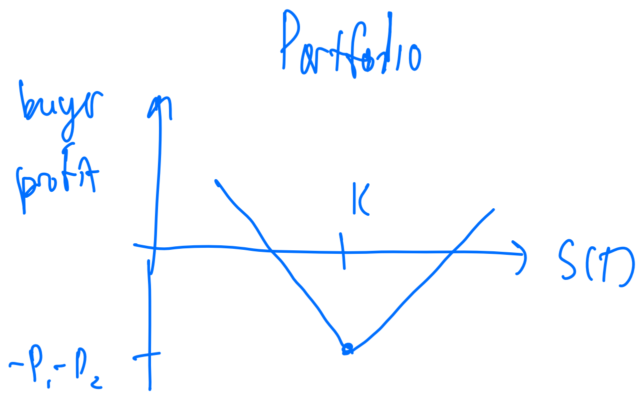
To more intuitively visualize such portfolio constructions, one typically draws a *profit* diagram, which vertically shifts the payoff diagram by the (net) premium (usually the future value thereof).

Europeen call premium of P_1

Europeen put premium of P_2







References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.