Math 5760/6890: Introduction to Mathematical Finance
Geometric Brownian Motion and SDE's

See Petters and Dong 2016, Sections 6.7-6.8

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

November 21, 2023

Brownian motion and stochastic integration

We have introduced the stochastic process Brownian motion, $B_t = B(t)$.

- $P(B(t) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n + 1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t - s)$.

Brownian motion and stochastic integration

We have introduced the stochastic process Brownian motion, $B_t = B(t)$.

- $P(B(t) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.

Using this, we have defined the Itô integral:

$$\int_0^T f(t) \mathrm{d}B_t = \lim_{n \uparrow \infty} \sum_{j=1}^{n} f(t_{j-1})(B(t_j) - B(t_{j-1})), \qquad t_j = \frac{jT}{n}.$$

Brownian motion and stochastic integration

We have introduced the stochastic process Brownian motion, $B_t = B(t)$.

- $P(B(t) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.

Using this, we have defined the Itô integral:

$$\int_0^T f(t) \mathrm{d}B_t = \lim_{n \uparrow \infty} \sum_{j=1}^{n} f(t_{j-1})(B(t_j) - B(t_{j-1})), \qquad t_j = \frac{jT}{n}.$$

The Itô integral can be used to notationally define differentials and (stochastic) differential equations:

$$X_T := \int_0^T f(t) \mathrm{d}B_t \quad \Longleftrightarrow \quad \mathrm{d}X_t = f(t) \mathrm{d}B_t.$$

$$\frac{dX_t}{dB_t} = f(t)$$

It is this differential notation that we will mostly exercise moving forward.

$$\frac{d}{dt} \int_a^t f(x)\,dx = f(t) \qquad \int_a^b f(x)\,dx = F(b) - F(a)$$

Some examples

Here are some examples of SDE's:

- Let $S_t = \mu t + \sigma B_t$, where B_t is a standard Brownian motion. To determine an SDE, note that by definition:

  $$B_t = \lim_{n\uparrow\infty} B_t = \lim_{n\uparrow\infty} \sum_{j=1}^{n} (B_j - B_{j-1}) = \int_0^T 1 \mathrm{d}B_t,$$

  i.e.,

  $$\mathrm{d}B_t = \mathrm{d}B_t.$$

  Combining this with linearity of the Itô integral and our usual understanding of the deterministic differential, we conclude:

  $$S_T = \int_0^T \mu \mathrm{d}t + \int_0^T \sigma \mathrm{d}B_t \quad \Longrightarrow \quad \mathrm{d}S_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t.$$

Some examples

Here are some examples of SDE's:

- Let $S_t = \mu t + \sigma B_t$, where B_t is a standard Brownian motion. To determine an SDE, note that by definition:

$$B_t = \lim_{n\uparrow\infty} B_t = \lim_{n\uparrow\infty} \sum_{j=1}^{n} (B_j - B_{j-1}) = \int_0^T 1 \mathrm{d}B_t,$$

i.e.,

$$\mathrm{d}B_t = \mathrm{d}B_t.$$

Combining this with linearity of the Itô integral and our usual understanding of the deterministic differential, we conclude:

$$S_T = \int_0^T \mu \mathrm{d}t + \int_0^T \sigma \mathrm{d}B_t \quad \Longrightarrow \quad \mathrm{d}S_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t.$$

- From last time:

$$\mathrm{d}(B_t^2) = \mathrm{d}t + 2B_t \mathrm{d}B_t$$

Itô processes and diffusion

Stochastic processes that are driven by Brownian motion have special terminology:

Suppose X_t is a stochastic process satisfying,

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t,$$

where $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are deterministic functions ("drift" and "volatility", respectively).

Then X_t is called an *Itô process*.

Itô processes and diffusion

Stochastic processes that are driven by Brownian motion have special terminology:

Suppose X_t is a stochastic process satisfying,

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t,$$

where $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are deterministic functions ("drift" and "volatility", respectively).

Then X_t is called an *Itô process*.

If $\mu = \mu(\cdot)$ and $\sigma = \sigma(\cdot)$, i.e.,

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

then X_t is an *Itô diffusion*.

Why the difference? Recall: $y' = y$, $y' = ty$ have different behavior.

Change of variables

A particularly useful tool is a change-of-variable result: Suppose a function satisfies,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu(x;t)$$

What differential equation does $y := f(x,t)$ satisfy?

$$\frac{dy}{dt} = \frac{d}{dt} f(x,t) = \frac{\partial f}{\partial t}(x,t) + \frac{\partial f}{\partial x}(x,t) \cdot \frac{dx}{dt}$$

chain rule

$$= \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x}$$

Change of variables

A particularly useful tool is a change-of-variable result: Suppose a function satisfies,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu(x; t)$$

What differential equation does $y := f(x, t)$ satisfy?

Through a simple application of the chain rule, we obtain:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} x'(t) \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu(\cancel{y}\,x; t) \end{aligned}$$

Change of variables

A particularly useful tool is a change-of-variable result: Suppose a function satisfies,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu(x;t)$$

What differential equation does $y := f(x,t)$ satisfy?

Through a simple application of the chain rule, we obtain:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} x'(t) \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu(u;t)\end{aligned}$$

If X_t is a trivial Itô process:

$$\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t, \qquad (\sigma \equiv 0)$$

then the standard chain rule would apply for $Y_t = f(X_t,t)$:

$$\mathrm{d}Y_t = \left(\frac{\partial f}{\partial t}(X_t,t) + \mu(X_t,t)\frac{\partial f}{\partial x}(X_t,t)\right)\mathrm{d}t.$$

But this does not apply for $\sigma \neq 0$.

Itô's Lemma

The corresponding chain rule-like result that includes the volatility is the following.

Lemma (Itô's Lemma)

Let X_t be an Itô process:

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t.$$

Define $Y_t = f(X_t, t)$. Then Y_t is an Itô process, and satisfies the SDE,

$$\mathrm{d}Y_t = \left(\frac{\partial f}{\partial t}(X_t, t) + \mu(X_t, t) \frac{\partial f}{\partial x}(X_t, t) + \frac{\sigma^2(X_t, t)}{2} \frac{\partial^2 f}{\partial x^2}(X_t, t) \right) \mathrm{d}t$$
$$+ \sigma(X_t, t) \frac{\partial f}{\partial x}(X_t, t) \mathrm{d}B_t$$

E.g.: $dB_t = dB_t$ $(\mu = 0, \sigma = 1)$

$f(x, t) = x^2$

$Y_t = f(B_t, t) = B_t^2$

Itô's Lemma: $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial^2 f}{\partial x^2} = 2$

$\Rightarrow dY_t = \frac{1^2}{2} \cdot 2\, dt + 1 \cdot 2 B_t\, dB_t$

$= dt + 2 B_t\, dB_t$

Itô's Lemma

The corresponding chain rule-like result that includes the volatility is the following.

Lemma (Itô's Lemma)

Let X_t be an Itô process:

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t.$$

Define $Y_t = f(X_t, t)$. Then Y_t is an Itô process, and satisfies the SDE,

$$\mathrm{d}Y_t = \left(\frac{\partial f}{\partial t}(X_t, t) + \mu(X_t, t)\frac{\partial f}{\partial x}(X_t, t) + \frac{\sigma^2(X_t, t)}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)\right)\mathrm{d}t + \sigma(X_t, t)\frac{\partial f}{\partial x}(X_t, t)\mathrm{d}B_t$$

More compactly: if we drop the explicit notational dependence on t, X_t, and use f_t, f_x, f_{xx} to denote partial derivatives then:

$$\mathrm{d}Y = \mu_Y \mathrm{d}t + \sigma_Y \mathrm{d}B,$$

with

$$\mu_Y = f_t + \mu f_x + \frac{\sigma^2}{2} f_{xx}, \qquad \sigma_Y = \sigma f_x.$$

Some intuition

Most of Itô's lemma is immediately motivated from deterministic calculus.

I.e., from the standard chain rule on deterministic quantities:

$$y(t) = f(x(t), t), \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(u; t),$$

we already expect that:

$$\begin{aligned} \mathrm{d}Y_t &= f_t \mathrm{d}t + f_x \left(\mu \mathrm{d}t + \sigma \mathrm{d}B\right) \\ &= \left(f_t + \mu f_x\right) \mathrm{d}t + \sigma f_x \mathrm{d}B. \end{aligned}$$

But in Itô's lemma, there is an additional $\frac{\sigma^2}{2} f_{xx} \mathrm{d}t$ term.

Some intuition

Most of Itô's lemma is immediately motivated from deterministic calculus.

I.e., from the standard chain rule on deterministic quantities:

$$y(t) = f(x(t), t), \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(u; t),$$

we already expect that:

$$\begin{aligned} \mathrm{d}Y_t &= f_t \mathrm{d}t + f_x \left(\mu \mathrm{d}t + \sigma \mathrm{d}B\right) \\ &= \left(f_t + \mu f_x\right) \mathrm{d}t + \sigma f_x \mathrm{d}B. \end{aligned}$$

But in Itô's lemma, there is an additional $\frac{\sigma^2}{2} f_{xx} \mathrm{d}t$ term.

This term arises because $[B]_t = t$, or $(\mathrm{d}B_t)^2 = \mathrm{d}t$:

$$\begin{aligned} f(X(t+\Delta t), t+\Delta t) - f(X(t), t) \approx & f_t \Delta t + (X(t+\Delta t) - X(t)) f_x \\ & + \frac{1}{2}(X(t+\Delta t) - X)^2 f_{xx} + \ldots \end{aligned}$$

The first-order derivatives yield what we already expect. The second order term yields,

$$\frac{1}{2}(X(t+\Delta t) - X)^2 f_{xx} \sim \frac{1}{2}\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right)^2 f_{xx} \sim \frac{1}{2}\sigma^2 f_{xx} (\mathrm{d}B_t)^2 + \ldots = \frac{1}{2}\sigma^2 f_{xx} \mathrm{d}t + \ldots$$

Some utility of Itô's Lemma

Itô's Lemma is extremely useful in general, but some utility that is particularly useful for us is the ability to identify SDEs for processes.

Example

Recall that

$\mu = 1 \quad \sigma = 2B_t$

$$\mathrm{d}(B_t^2) = \mathrm{d}t + 2B_t \mathrm{d}B_t$$

Construct an SDE for $e^{(B_t^2)+t}$, and identify the drift and volatility functions.

$f(x,t) = e^{x+t} \implies f((B_t)^2, t) = e^{(B_t)^2+t} = Y_t$

$\frac{\partial f}{\partial t} = e^{x+t}, \quad \frac{\partial f}{\partial x} = e^{x+t}, \quad \frac{\partial^2 f}{\partial x^2} = e^{x+t}$

$dY = \underbrace{\left(f_t + \mu f_x + \frac{\sigma^2}{2} f_{xx}\right)}_{\mu_Y} dt + \underbrace{(\sigma f_x)}_{\sigma_Y} dB_t$

$$\Rightarrow \mu_Y = e^{(B_t)^2+t}\left[1 + 1 + \frac{4(B_t)^2}{2}\right] = Y_t\, 2\left(1 + (B_t)^2\right)$$

$$= Y_t \cdot 2 \cdot \left[1 + \log(Y_t) - t\right]$$

$$\sigma_Y = 2B_t\, e^{(B_t)^2+t}$$

$$= 2\sqrt{\log Y_t - t} \cdot Y_t$$

Geometric Brownian Motion

Back to securities, we assumed availability of continuous-time drift and volatility (μ, σ).

We've seen that a reasonable stochastic model for the log-return $L(t)$ is,

$$L_t = \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion.

Geometric Brownian Motion

Back to securities, we assumed availability of continuous-time drift and volatility (μ, σ).

We've seen that a reasonable stochastic model for the log-return $L(t)$ is,

$$L_t = \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion.

We then expect that a reasonable model for the security price is,

$$S_t = S_0 e^{L_t} = S_0 e^{\mu t + \sigma B_t}.$$

What kind of SDE does S_t satisfy? $dS_t = ?$

$f(x,t) = e^x$

$\frac{S_t}{S_0} = f(L_t, t) = e^{L_t}$

$\frac{\partial f}{\partial t} = 0$

$\frac{\partial f}{\partial x} = e^x$

$\frac{\partial^2 f}{\partial x^2} = e^x$

Itô's lemma: $dS_t = \left[\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right]dt + \sigma \frac{\partial f}{\partial x}\, dB_t$

$$dS_t = \left[\mu e^{L_t} + \frac{\sigma^2}{2} e^{L_t}\right] dt + \sigma e^{L_t} dB_t$$

$$dS_t = \frac{1}{S_0}\left[\left(\mu + \frac{\sigma^2}{2}\right) S_t \, dt + \sigma S_t \, dB_t\right]$$

– or – :

$$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t \, dt + \sigma S_t \, dB_t$$

$$S(0) = S_0$$

Geometric Brownian Motion

Back to securities, we assumed availability of continuous-time drift and volatility (μ, σ).

We've seen that a reasonable stochastic model for the log-return $L(t)$ is,

$$L_t = \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion.

We then expect that a reasonable model for the security price is,

$$S_t = S_0 e^{L_t} = S_0 e^{\mu t + \sigma B_t}.$$

What kind of SDE does S_t satisfy?

Definition

Let μ, σ be constants, $\sigma > 0$. With B_t a standard Brownian motion, suppose S_t is a stochastic process defined by,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad S(t=0) = S_0.$$

Then S_t is a **Geometric Brownian Motion** with drift and volatility (μ, σ).

Geometric Brownian Motion

Back to securities, we assumed availability of continuous-time drift and volatility (μ, σ).

We've seen that a reasonable stochastic model for the log-return $L(t)$ is,

$$L_t = \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion.

We then expect that a reasonable model for the security price is,

$$S_t = S_0 e^{L_t} = S_0 e^{\mu t + \sigma B_t}.$$

What kind of SDE does S_t satisfy?

Definition

Let μ, σ be constants, $\sigma > 0$. With B_t a standard Brownian motion, suppose S_t is a stochastic process defined by,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad S(t=0) = S_0.$$

Then S_t is a **Geometric Brownian Motion** with drift and volatility (μ, σ).

Note that S_t as defined above corresponds to the process,

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t)},$$

which is a $\mathrm{lognormal}\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$ random variable.

The gist and summary

We set out to construct a continuous-time analogue of the binomial tree model for securities.

- Geometric Brownian motion, i.e., a stochastic process satisfying an SDE of the form,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad + \quad S(0) = S_0$$

  is our model.

The gist and summary

We set out to construct a continuous-time analogue of the binomial tree model for securities.

- Geometric Brownian motion, i.e., a stochastic process satisfying an SDE of the form,

  $$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t,$$

  is our model.
- This is a stochastic model for an asset: given historical data, we can simulate future trajectories of stock prices by numerically discretizing this model.

The gist and summary

We set out to construct a continuous-time analogue of the binomial tree model for securities.

- Geometric Brownian motion, i.e., a stochastic process satisfying an SDE of the form,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t,$$

  is our model.
- This is a stochastic model for an asset: given historical data, we can simulate future trajectories of stock prices by numerically discretizing this model.
- This model can be viewed as the continuous-time limit of the CRR model for S_n: the price of S_n in the CRR model evolves according to *geometric* increments. Geometric Brownian motion follows a similar principle (as the SDE reveals).

The gist and summary

We set out to construct a continuous-time analogue of the binomial tree model for securities.

- Geometric Brownian motion, i.e., a stochastic process satisfying an SDE of the form,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t,$$

  is our model.
- This is a stochastic model for an asset: given historical data, we can simulate future trajectories of stock prices by numerically discretizing this model.
- This model can be viewed as the continuous-time limit of the CRR model for S_n: the price of S_n in the CRR model evolves according to *geometric* increments. Geometric Brownian motion follows a similar principle (as the SDE reveals).
- One nice thing about the SDE formulation: it's ok if μ or σ vary with time, or even depend on S_t. This model is *flexible*.

Simulating SDEs

How does one numerically simulate SDE paths? It's not quite the same as coin flips. The model is,

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad S(t=0) = S_0.$$

Given an equispaced set of times,

$$t_j = hj, \qquad h > 0,$$

and for notational ease setting $S_{t_j} = S_j$, then how do we generate a trajectory?

Simulating SDEs

How does one numerically simulate SDE paths? It's not quite the same as coin flips. The model is,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad S(t=0) = S_0.$$

Given an equispaced set of times,

$$t_j = hj, \qquad h > 0,$$

and for notational ease setting $S_{t_j} = S_j$, then how do we generate a trajectory? The discrete differential form of the SDE provides one possible simple answer:

$$S_{j+1} - S_j = \mu S_j \underbrace{(t_{j+1} - t_j)}_{h} + \sigma S_j \underbrace{(B_{j+1} - B_j)}_{\text{Brownian motion increment}}.$$

$\mathcal{N}(0, h)$

Simulating SDEs

How does one numerically simulate SDE paths? It's not quite the same as coin flips. The model is,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad\qquad S(t=0) = S_0.$$

Given an equispaced set of times,

$$t_j = hj, \qquad\qquad h > 0,$$

and for notational ease setting $S_{t_j} = S_j$, then how do we generate a trajectory? The discrete differential form of the SDE provides one possible simple answer:

$$S_{j+1} - S_j = \mu S_j (t_{j+1} - t_j) + \sigma S_j (B_{j+1} - B_j).$$

But $t_{j+1} - t_j = h$, and $B_{j+1} - B_j \sim \mathcal{N}(0, h)$, so this scheme can be written as,

$$S_{j+1} = S_j + \mu h S_j + \sigma S_j \sqrt{h} Z, \qquad\qquad Z \sim \mathcal{N}(0,1),$$

with S_0 given and fixed.

This scheme is called the **Euler-Maruyama method**.
(This is exactly how Brownian motion figures were generated on previous slides.)

A digression on SDEs

We won't really use SDE's in more complicated situations than we've covered.

But SDEs are **enormously** useful in various non-finance contexts.

A digression on SDEs

We won't really use SDE's in more complicated situations than we've covered.

But SDEs are **enormously** useful in various non-finance contexts.

Here is one example: Abstractly, an SDE allows us to determine a target behavior through the *drift*, and the *volatility* provides a mechanism to generate randomness around the target.

Around the target is key: of course we can generate completely random things that might not make sense.

A digression on SDEs

We won't really use SDE's in more complicated situations than we've covered.

But SDEs are **enormously** useful in various non-finance contexts.

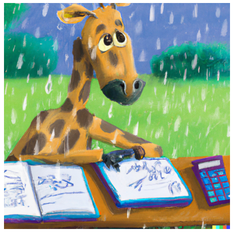
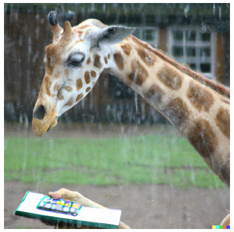
Here is one example: Abstractly, an SDE allows us to determine a target behavior through the *drift*, and the *volatility* provides a mechanism to generate randomness around the target.

<u>Around the target</u> is key: of course we can generate completely random things that might not make sense.

Suppose we tried to generate random images of objects or scenarios: Abstractly, what this means is that there should be a pipeline that accomplishes:

"A giraffe completing mathematical finance homework in the rain" $\rightarrow$ "Drift" μ $\rightarrow$ Generate SDE trajectories

That should generate an output like:


SDEs for images

Or even more abstractly, the input "target" can be an image itself:

And the output could be SDE-based stochastic diffusions of the input:

Diffusion maps

This methodology is called creating a **diffusion map**, although there are many variants.

And this is exactly what certain **generative AI** software does, in particular image-based AI generators.

SDEs are a large part of the mathematics that underlie these deep learning-based models.

Diffusion maps

This methodology is called creating a **diffusion map**, although there are many variants.

And this is exactly what certain **generative AI** software does, in particular image-based AI generators.

SDEs are a large part of the mathematics that underlie these deep learning-based models.

The key, very difficult problem is a proper learning of the drift: one has to hit the right target in image space.
(Of course one should also add "noise" in the right ways.)

This is why most diffusion-based generative AI models require lots of data: training the drift and volatility so that the SDE evolution generates meanginful outputs is very hard.
(The figures on the previous slides were generated using DALL-E.)

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.