Math 5760/6890: Introduction to Mathematical Finance
A (rather non-technical) intro to Stochastic differential equations

See Petters and Dong 2016, Sections 6.6-6.7

Akil Narayan[1]

[1] **Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute**
University of Utah

November 16, 2023

Brownian motion

We have introduced Brownian motion, a stochastic process $B(t)$ having the properties:

- $P(B(t) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.
- With probability 1, the sample path $B(\cdot, \omega)$ is differentiable nowhere.
- For any interval $I \subset [0, \infty)$ of finite length, with probability 1, the sample path $B(\cdot, \omega)$ has infinite “variation" on I.
- B has Markovian structure: Fix $s > 0$ and define $A(t) := B(t+s) - B(s)$. Then $A(t)$ and $B(t)$ have the same distribution, and in particular A is a standard Brownian motion.
- Sample paths of Brownian motion are self-similar/fractals. In particular, for any $c > 0$, the process $\frac{1}{c} B(c^2 t)$ is a standard Brownian motion.

The first few of these are defining properties of Brownian motion, while the latter ones are consequences of the definition.

Brownian motion

We have introduced Brownian motion, a stochastic process $B(t)$ having the properties:

- $P(B(t) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.
- With probability 1, the sample path $B(\cdot, \omega)$ is differentiable nowhere.
- For any interval $I \subset [0, \infty)$ of finite length, with probability 1, the sample path $B(\cdot, \omega)$ has infinite "variation" on I.
- B has Markovian structure: Fix $s > 0$ and define $A(t) := B(t+s) - B(s)$. Then $A(t)$ and $B(t)$ have the same distribution, and in particular A is a standard Brownian motion.
- Sample paths of Brownian motion are self-similar/fractals. In particular, for any $c > 0$, the process $\frac{1}{c} B(c^2 t)$ is a standard Brownian motion.

The first few of these are defining properties of Brownian motion, while the latter ones are consequences of the definition.

Our next goal: Stochastic calculus.
The standard way this is introduced is first to discuss stochastic integration, followed by concepts of stochastic differentiation.

Why stochastic calculus?

Calculus is the language of (instantaneous) *change* – modeling change is foundational in many fields, including finance.

Recall that an annual interest rate r results in a time $T = 1$ future value of an amount S_0 according to

$$S_1 = S_0 \left(1 + \frac{r}{n}\right)^n,$$

where interest is compounded n times over a single year.

Why stochastic calculus?

Calculus is the language of (instantaneous) *change* – modeling change is foundational in many fields, including finance.

Recall that an annual interest rate r results in a time $T = 1$ future value of an amount S_0 according to

$$S_1 = S_0 \left(1 + \frac{r}{n}\right)^n,$$

where interest is compounded n times over a single year.

One can also imagine using continuous compounding (taking $n \to \infty$), which results in the amount,

$$S_1 = S_0 e^r,$$

or for an arbitrary terminal time $t > 0$:

$$S_t = S_0 e^{rt}.$$

Why stochastic calculus?

Calculus is the language of (instantaneous) *change* – modeling change is foundational in many fields, including finance.

Recall that an annual interest rate r results in a time $T = 1$ future value of an amount S_0 according to

$$S_1 = S_0 \left(1 + \frac{r}{n}\right)^n,$$

where interest is compounded n times over a single year.

One can also imagine using continuous compounding (taking $n \to \infty$), which results in the amount,

$$S_1 = S_0 e^r,$$

or for an arbitrary terminal time $t > 0$:

$$S_t = S_0 e^{rt}.$$

The more constructive way to view this result is as a differential equation:

$$S'(t) = rS(t), \qquad S(0) = S_0.$$

I.e., we can write this as a calculus/differential equations problem.

Why stochastic calculus

What use is calculus for this problem?

Suppose I wanted to know the future value of an asset at time $T = 1$, where the interest rate increases linearly from r to $2r$ over the $[0, T]$ time interval.

Why stochastic calculus

What use is calculus for this problem?

Suppose I wanted to know the future value of an asset at time $T = 1$, where the interest rate increases linearly from r to $2r$ over the $[0, T]$ time interval.

With calculus, the formulation of this is easy:

$$S'(t) = r(1+t)S(t), \qquad S(0) = S_0.$$

Why stochastic calculus

What use is calculus for this problem?

Suppose I wanted to know the future value of an asset at time $T = 1$, where the interest rate increases linearly from r to $2r$ over the $[0, T]$ time interval.

With calculus, the formulation of this is easy:

$$S'(t) = r(1+t)S(t), \qquad S(0) = S_0.$$

And if we learn enough calculus, the solution is easy, too:

$$S(t) = S_0 \exp\left(rt + \frac{rt^2}{2}\right).$$

Why stochastic calculus

What use is calculus for this problem?

Suppose I wanted to know the future value of an asset at time $T = 1$, where the interest rate increases linearly from r to $2r$ over the $[0, T]$ time interval.

With calculus, the formulation of this is easy:

$$S'(t) = r(1+t)S(t), \qquad S(0) = S_0.$$

And if we learn enough calculus, the solution is easy, too:

$$S(t) = S_0 \exp\left(rt + \frac{rt^2}{2}\right).$$

The same principles are true for our probabilistic valuation models:

- It rather technical to understand model dependence on complicated drift and volatility for discrete-time models.
- Continuous time is a convenient standardization: it's easy to specialize a continuous-time model to discrete-time trading, but it's harder to specialize a discrete-time model to discrete-time with a different period.
- Calculus provides technical intuition: it's much easier to understand behavior of calculus models than it is to understand the corresponding behavior of discrete-time models.

Why stochastic calculus

What use is calculus for this problem?

Suppose I wanted to know the future value of an asset at time $T = 1$, where the interest rate increases linearly from r to $2r$ over the $[0, T]$ time interval.

With calculus, the formulation of this is easy:

$$S'(t) = r(1+t)S(t), \qquad S(0) = S_0.$$

And if we learn enough calculus, the solution is easy, too:

$$S(t) = S_0 \exp\left(rt + \frac{rt^2}{2}\right).$$

The same principles are true for our probabilistic valuation models:

- It rather technical to understand model dependence on complicated drift and volatility for discrete-time models.
- Continuous time is a convenient standardization: it's easy to specialize a continuous-time model to discrete-time trading, but it's harder to specialize a discrete-time model to discrete-time with a different period.
- Calculus provides technical intuition: it's much easier to understand behavior of calculus models than it is to understand the corresponding behavior of discrete-time models.
- The downside: stochastic calculus is <u>much</u> more technical and advanced than standard calculus.

Quadratic variation

Our first step in stochastic calculus is integration, from which we will introduce dual concepts of differentiation.

The rationale for this is practical: since Brownian motion is nowhere differentiable, we cannot really hope to use standard notions of differentiation to understand the evolution of Brownian motion.

Quadratic variation

Our first step in stochastic calculus is integration, from which we will introduce dual concepts of differentiation.

The rationale for this is practical: since Brownian motion is nowhere differentiable, we cannot really hope to use standard notions of differentiation to understand the evolution of Brownian motion.

Our first step in the direction of stochastic integration is to introduce quadratic variation.

Definition

Let $X(t)$ and $Y(t)$ be two stochastic processes. The quadratic covariation of X_t and Y_t on the interval $[0, T]$ is the stochastic process given by,

$$[X, Y]_T := \lim_{n \to \infty} \sum_{j=1}^{n} \left(X_j - X_{j-1}\right)\left(Y_j - Y_{j-1}\right), \qquad X_j := X(t_j) = X\left(j \frac{T}{n}\right).$$

The quadratic variation of X over the same interval is,

$$[X]_T := [X, X]_T.$$

Quadratic variation

Our first step in stochastic calculus is integration, from which we will introduce dual concepts of differentiation.

The rationale for this is practical: since Brownian motion is nowhere differentiable, we cannot really hope to use standard notions of differentiation to understand the evolution of Brownian motion.

Our first step in the direction of stochastic integration is to introduce quadratic variation.

Definition

Let $X(t)$ and $Y(t)$ be two stochastic processes. The quadratic covariation of X_t and Y_t on the interval $[0, T]$ is the stochastic process given by,

$$[X, Y]_T := \lim_{n \to \infty} \sum_{j=1}^{n} (X_j - X_{j-1})(Y_j - Y_{j-1}), \qquad X_j := X(t_j) = X\left(j\frac{T}{n}\right).$$

The quadratic variation of X over the same interval is,

$$[X]_T := [X, X]_T.$$

There is a lot of use in understanding quadratic (co)variation, but for us the following simple fact suffices: If B is a standard Brownian motion, then

$$[B]_T = T$$

Why? $[B]_T \approx \sum_{j=1}^{n} (B_{j+1} - B_j)^2$

$B_{j+1} - B_j \sim N(0, T/n)$ ← $t_{j+1} - t_j$

So $(B_{j+1} - B_j)^2$ "$\approx$" T/n

$\Rightarrow [B]_T \approx \sum_{j=1}^{n} T/n = T$ (HW)

A taste of differentials

The fact that

$$[B, B]_T = T$$

suggests some notation that we will use soon.

First, note that this is true for T very small, and this along with the Markovian property of B suggsets that we have the approximation,

$$(B(t + \Delta t) - B(t))^2 \approx \Delta t.$$

A taste of differentials

The fact that

$$[B, B]_T = T$$

suggests some notation that we will use soon.

First, note that this is true for T very small, and this along with the Markovian property of B suggsets that we have the approximation,

$$(B(t + \Delta t) - B(t))^2 \approx \Delta t.$$

An immediately suggestive notation for the above statement in the limit of infinitesimal Δt is,

$$(dB)^2 = dt.$$

A taste of differentials

The fact that

$$[B, B]_T = T$$

suggests some notation that we will use soon.

First, note that this is true for T very small, and this along with the Markovian property of B suggsets that we have the approximation,

$$(B(t + \Delta t) - B(t))^2 \approx \Delta t.$$

An immediately suggestive notation for the above statement in the limit of infinitesimal Δt is,

$$(dB)^2 = dt.$$

To emphasize that B depends on t, one typically uses a subscript:

$$(dB_t)^2 = dt.$$

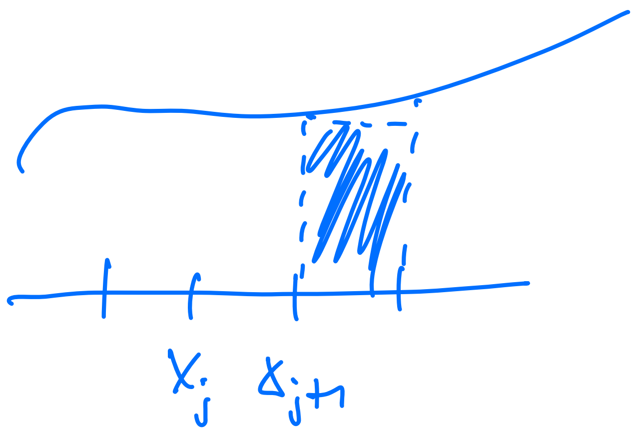
Note that for stochastic processes, subscript notation is *not* differentiation.

Stochastic integration

We can now introduce integration: our goal will be to integrate *with respect to* Brownian motion.

To gain some understanding of what this means, recall Riemann integration: if $f : [0,1] \to \mathbb{R}$, then

$$\int_0^1 f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1}), \qquad x_j := \frac{j}{n}.$$



Stochastic integration

We can now introduce integration: our goal will be to integrate *with respect to* Brownian motion.

To gain some understanding of what this means, recall Riemann integration: if $f : [0,1] \to \mathbb{R}$, then

$$\int_0^1 f(x) \mathrm{d}x = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1}), \qquad x_j := \frac{j}{n}.$$

This also works just fine if you want to integrate with respect to a different variable:

$$u = x^2 \implies \int_0^1 f(x) \mathrm{d}u(x) := \lim_{n \to \infty} \sum_{j=1}^{n} f(x_{j-1})(u_j - u_{j-1}), \qquad u_j := u(x_j).$$

??

$$\sum_{j=1}^{n} f(x_j)\left(x_j^2 - x_{j-1}^2\right) = \sum_{j=1}^{n} f(x_j)(x_{j+1} + x_j)(x_{j+1} - x_j)$$

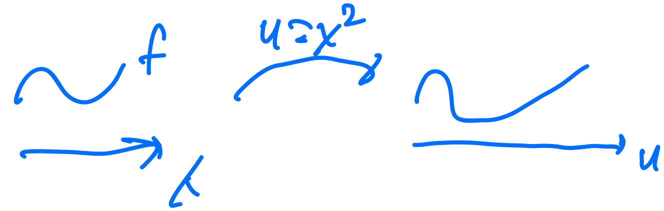
$$u = x^2 \implies du = 2x\,dx$$

$$\int_0^1 f(x)\,du(x) = \int_0^1 f(x)\,2x\,dx$$

$$\approx \sum_{j=1}^{n} f(x_j)\,2x_j\,(x_{j+1} - x_j)$$

$$\xrightarrow{n \uparrow \infty} \int_0^1 f(x)\,2x\,dx$$

Stochastic integration



We can now introduce integration: our goal will be to integrate *with respect to* Brownian motion.

To gain some understanding of what this means, recall Riemann integration: if $f : [0, 1] \to \mathbb{R}$, then

$$\int_0^1 f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1}), \qquad x_j := \frac{j}{n}.$$

This also works just fine if you want to integrate with respect to a different variable:

$$u = x^2 \implies \int_0^1 f(x)\mathrm{d}u(x) := \lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j-1})(u_j - u_{j-1}), \qquad u_j := u(x_j).$$

This inspires the following definition, which is the cornerstone of stochastic calculus:

Definition (Itô Integral)

Let $f(t) = f_t$ be a stochastic process (possibly also a deterministic function). Then we define,

$$\int_0^T f_t \mathrm{d}B_t := \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1}) \left(B(t_j) - B(t_{j-1})\right).$$

where B_t is a standard Brownian motion. This is called an **Itô Integral**.

Stochastic differentials

The stochastic process,

$$X_t = \int_0^T f_t \mathrm{d}B_t$$

is called a process that is *driven by Brownian motion*.

Stochastic differentials

The stochastic process,

$$X_t = \int_0^T f_t \mathrm{d}B_t$$

is called a process that is *driven by Brownian motion*.

Because Riemann integration statements can be written in terms of differentials, then we are tempted to use differential notation for the above expression:

$$\frac{dX_t}{dB_t} \not= f_t \quad \Longleftarrow \quad dX_t = f_t \mathrm{d}B_t \qquad \Longleftrightarrow \qquad X_t = \int_0^T f_t \mathrm{d}B_t$$

It is important to understand that the differential notation is just notation! The real meaning is an integral statement: a rigorous statement involving standard differentials and derivatives is not possible!

However: this is our first example of a *stochastic* differential equation (SDE).

Stochastic differentials

The stochastic process,

$$X_t = \int_0^T f_t \mathrm{d}B_t$$

is called a process that is *driven by Brownian motion*.

Because Riemann integration statements can be written in terms of differentials, then we are tempted to use differential notation for the above expression:

$$dX_t = f_t \mathrm{d}B_t \quad \Longleftrightarrow \quad X_t = \int_0^T f_t \mathrm{d}B_t$$

It is important to understand that the differential notation is just notation! The real meaning is an integral statement: a rigorous statement involving standard differentials and derivatives is not possible!

However: this is our first example of a *stochastic* differential equation (SDE).

Example

Let B_t be a standard Brownian motion. Some direct computations involving B_t^2 allow us both to compute $\int_0^t B_t \mathrm{d}B_t$, and to derive an SDE for B_t^2.

$$B_t^2 = B_t^2 - B_0^2 = \sum_{j=1}^{n} \left(B_j^2 - B_{j-1}^2\right), \qquad B_j = B(t_j), \quad t_j = t/n, \quad n \in \mathbb{N}$$

$$= \sum_{j=1}^{n} (B_j - B_{j-1})(B_j + B_{j-1})$$

$$= \sum_{j=1}^{n} (B_j - B_{j-1})(B_j - B_{j-1} + 2B_{j-1})$$

$$= \sum_{j=1}^{n} (B_j - B_{j-1})^2 + 2\sum_{j=1}^{n} B_{j-1}(B_j - B_{j-1})$$

$$\xrightarrow{n \uparrow \infty} [B]_t + 2\int_0^t B_t \, dB_t$$

Why is $[B]_t = t$?

1.) $\mathbb{E} \sum_{j=1}^{n} (B_j - B_{j-1})^2 = t$

2.) $\mathrm{Var} \sum_{j=1}^{n} (B_j - B_{j-1})^2 \rightsquigarrow 0$ as $n \nearrow \infty$

$$\Rightarrow B_t^2 = t + 2\int_0^t B_t \, dB_t$$

a) $\int_0^t B_t \, dB_t = \frac{1}{2} B_t^2 - t/2$

b) Define $X_t = B_t^2$

$$\Longrightarrow dX_t = dt + 2B_t \, dB_t \qquad X_0 = 0 \text{ wp } 1$$

(SDE for B_t^2 !)

Other SDE's?

B_t $\quad (= L_t$ for $\mu = 0, \sigma = 1)$

$dB_t = dB_t \qquad B_t = \int_0^t dB_t$

$\mu \neq 0, \sigma \neq 1$

$$L(t) = \mu t + \sigma B_t$$

$$\Rightarrow dL_t = \mu dt + \sigma dB_t$$

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.