Math 5760/6890: Introduction to Mathematical Finance
A (rather non-technical) primer on Stochastic processes

See Petters and Dong 2016, Sections 6.1-6.6

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A brief intro to stochastic processes

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To get there we'll go through some abstract formalization:

Definition

Let Ω be (probabilistic) event space, and let $\mathcal{I}$ be a set.
A stochastic process $X = X(t, \omega)$ is a map,

$$X : \mathcal{I} \times \Omega \to \mathbb{R}, \qquad X(t, \omega) \in \mathbb{R} \text{ for every } t \in \mathcal{I} \text{ and } \omega \in \Omega.$$

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The set $\mathcal{I}$ is called the *index set* of the process X, and $\mathbb{R}$ is the *state space*.

For fixed $t \in \mathcal{I}$, $X(t, \cdot)$ is a scalar random variable (e.g., and has a distribution).

For fixed $\omega \in \Omega$, $X(\cdot, \omega)$ is a deterministic scalar-valued function on the domain $\mathcal{I}$.

Some examples

Example

Let $\boldsymbol{X} \in \mathbb{R}^N$ be a random vector. (Say the unknown time-1 price in Markowitz portfolio analysis.)

Then $\boldsymbol{X}$ is a stochastic process with index set $\mathcal{I} = \{1, 2, \ldots, N\}$.

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Example

Consider an n-period Binomial tree model with parameters $(p, u, d) = (p_n, u_n, d_n)$ fixed. (E.g., via a CRR model.)

The asset prices at every period can be lumped into a vector $\boldsymbol{S} := (S_0, S_1, \ldots, S_n)^T \in \mathbb{R}^{n+1}$.

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$\boldsymbol{S} := (S_0, S_1, \ldots, S_n)^T \in \mathbb{R}^{n+1}$.

Then $\boldsymbol{S}$ is a stochastic process with index set $\mathcal{I} = \{0, 1, \ldots, n\}$.

Continuum index sets

The previous examples had *discrete* index sets.

Of course, there is nothing stopping us from considering a continuous index set.

For example, if $S(t)$, $t \in [0, T]$ was the continuous-time model that we described from our $n \uparrow \infty$ limit of the CRR model, then S is a stochastic process with index set $\mathcal{I} = [0, T]$.

Recall: $S(t) \sim \text{lognormal}(\log S_0 + \mu t, \sigma^2 t)$

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When we have an index set that's a continuum, we can discussing some standard notions of functional analysis.

Definition

A realization of a stochastic process $X(\cdot,\omega)$ (for a fixed $\omega \in \Omega$) with a continuum index set $\mathcal{I}$ is said to have a **continuous sample path** if

$$\lim_{s \to t} X(s,\omega) = X(x,\omega), \qquad \text{for every } t \in \mathcal{I}.$$

This notion of continuity involves a single, fixed ω.

Continuous stochastic processes

A proper extension of continuity (or any other) property to "all ω" is more delicate.

Definition

Let X be a stochastic process with a continuum index set $\mathcal{I} = [0, \infty)$. Then X is **continuous** if

$$\widetilde{\Omega} := \left\{\omega \in \Omega \;\middle|\; X \text{ has a continuous sample path at } \omega\right\},$$

satisfies $P(\widetilde{\Omega}) = 1$.

Alternative languange: X is sample-path continuous, or almost surely continuous, or continuous with probability 1.

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Why not "for all ω"?

The mathematics of stochastic processes makes statements "with probability 1" the more natural statements to consider.

Asking for properties "for every ω" is too strong in the sense that asserting this limits our flexibility for analysis.

And since we really only care about probabilities of things, asking for statements in terms of probabilities is conceptually natural.

A stochastic process for securities

Our next major goal will be to identify and investigate a particular stochastic process that forms the foundation of mathematical finance.

To describe what kind of stochastic process we want, let's consider the log-return for our n-period discrete-time binomial tree model:

$$S_j = S(t_j), \qquad L_j = \log \frac{S_{j+1}}{S_j}, \qquad t_j = j h_n, \qquad h_n = \frac{T}{n}.$$

Recall that we model L_j through a coin flip. More precisely,

$$L_j \sim \log d_n + \log \frac{u_n}{d_n} X, \qquad X \sim \mathrm{Bernoulli}(p_n).$$

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Suppose we take $n \uparrow \infty$, and we define $L(t)$ as the *cumulative* log-return from time 0:

$$L(T) \text{``}=\text{''} \log S_0 + \lim_{n\uparrow\infty} \sum_{j=1}^{n} L_j.$$

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This in principle defines a stochastic process $L(t)$ with index set $\mathcal{I} = [0, T]$. What properties should this process have?

A more explicit construction

Consider our real-world CRR tree, which assigns (p_n, u_n, d_n) as,

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right),$$

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Under this assumption, then

log u_n

$$L_j = \begin{cases} \sqrt{h_n}, & \text{with probability } \frac{1}{2}, \\ -\sqrt{h_n}, & \text{with probability } \frac{1}{2}, \end{cases}$$

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The goal now is to identify/construct a stochastic process $L(t)$ that:

- is consistent with what happens to $\log(S_n/S_0)$ when we take $n \uparrow \infty$
- has the properties we desire from the finance perspective

Desiderata

$S(t) \sim \text{lognormal}(\log S_0 + \mu t, \sigma^2 t)$

$L(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$

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There is essentially just one process satisfying these conditions: Brownian motion.

Brownian motion

Definition

With index set $\mathcal{I} = [0, \infty)$, a **standard Brownian motion/Wiener process** $B = B_t = B(t)$ is a stochastic process satisfying

1. $P(B(0) = 0) = 1$
2. B is continuous with probability 1
3. The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
4. For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t - s)$.

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- The last property implies that for every $s, t, h \geqslant 0$:

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- The four properties above are typically concisely referred to, respectively, as: $B(0) =$ ~~1~~ 0 with probability 1, B is sample-continuous with probability 1, B has independent increments, and B has (time-)stationary and normally-distributed increments.

Affine functions of Brownian motion

Let $B(t)$ be a standard Brownian motion, and let $b_0 \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma > 0$. Then the process

$$A(t) = b_0 + \mu t + \sigma B(t),$$

is a Brownian motion with drift and scaling:

- It has time-0 value b_0 with probability 1.
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Example

What distribution do the increments of $A(t)$ have?

$$A(s) - A(t) = \mu(s-t) + \sigma\underbrace{(B(s) - B(t))}_{\sim \mathcal{N}(0, |s-t|)} \sim \mathcal{N}\left(\mu(s-t), \sigma^2 |s-t|\right)$$

Other properties

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Why? $L_{j+1} - L_j \sim \begin{cases} \sqrt{h_n} & \text{w/prob } 1/2 \\ -\sqrt{h_n} & \text{w/prob } 1/2 \end{cases}$

$$\text{"}\frac{dL}{dt}\text{"} = \frac{L_{j+1} - L_j}{h_n} = \frac{\pm 1}{\sqrt{h_n}} \qquad \lim_{h_n \to 0} \text{"}\frac{dL}{dt}\text{"} \to \text{DNE}$$

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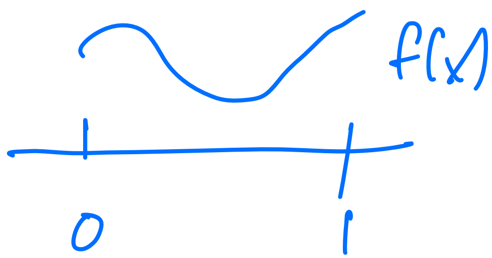
- With probability 1, the sample path $B(\cdot, \omega)$ is differentiable <u>nowhere</u>
- For any interval $I \subset [0, \infty)$ of finite length, with probability 1, the sample path $B(\cdot, \omega)$ has infinite "variation" on I.

$f(x)$, $x \in [0, 1]$

"variation": amount of change that f undergoes



variation: $\int_0^1 |f'(x)| \, dx$

$$\sum_j \frac{|f_{j+1} - f_j|}{\Delta x} \Delta x$$

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- B has Markovian structure: Fix $s > 0$ and define $A(t) := B(t + s) - B(s)$. Then $A(t)$ and $B(t)$ have the same distribution, and in particular A is a standard Brownian motion.

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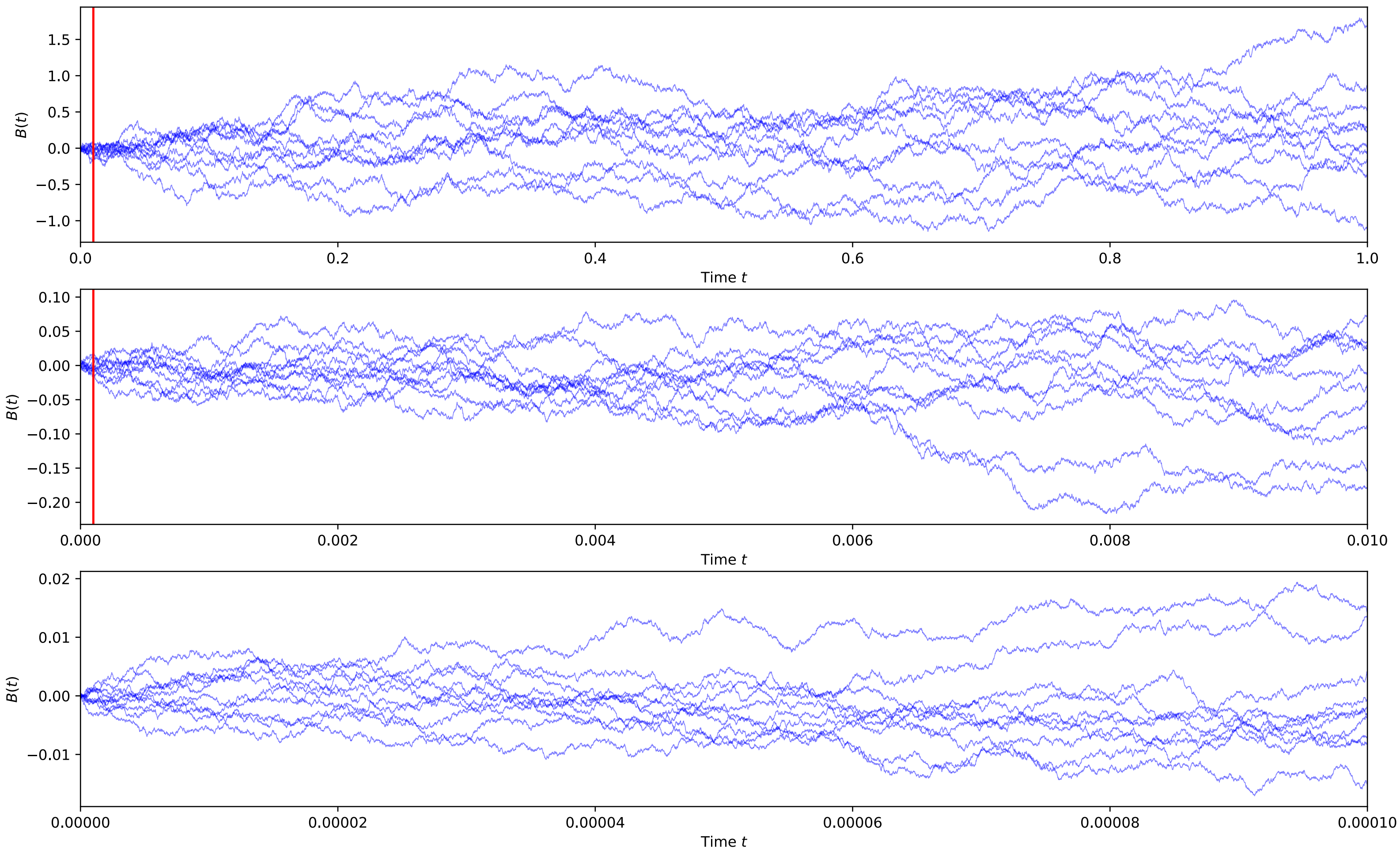
- Sample paths of Brownian motion are self-similar/fractals. In particular, for any $c > 0$, the process $\frac{1}{c}B(c^2 t)$ is a standard Brownian motion.

Self-similarity of Brownian paths



In relation to securities....

To close the loop: a standard Brownian motion $B(t)$ is exactly a continuous-time process whose properties line up with $L(t)$, our continuous-time limit of the binomial tree model.

In particular, if $(\mu, \sigma, S_0) = (0, 1, 1)$, then we will identify

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While in principle we are done in terms of identifying a mathematical model for $L(t)$, we have actually just begun to reap benefits from this model.

In particular, the fact that $B(t)$ has sample paths or *trajectories* suggests that there is an underlying time-evolution law.

Stochastic calculus is the appropriate language we'll use to explore such concepts.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.