Final Exam: Friday Dec 15, 8-10am "here"

Open notes/book/calc. exam

Problems: inspired by HW problems.

Material covered: whatever's on homeworks (+ projects)

Math 5760/6890: Introduction to Mathematical Finance
Continuous-time limits, I

See Petters and Dong 2016, Section 5.2, 5.3

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November 7, 2023

The binomial tree pricing and CRR models

We have modeled a security's price $S_j = S(t_j)$ via,

$$S_{j+1} = G_{j+1} S_j, \qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

From this model, we've concluded:

- $L := \log(S_n/S_0)$ is a scaled/shifted $\mathrm{Binomial}(n,p)$ random variable.
- $S_n = S_0 e^L$ is the exponential of a scaled/shifted Binomial random variable
- The triple (p, u, d) determines the distribution entirely.

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The CRR model places the following additional constraints on our standard Binomial tree model:

- Geometric symmetry of tree prices: $u = 1/d$
- The continuous-time limit of the expected log-return matches the real-world drift:
- The continuous-time limit of the variance of the log-return matches the real-world (squared) volatility:

This results (after some approximation) in the following *real-world CRR equations*:

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right).$$

The distribution of S_n

as $n \uparrow \infty$

What kind of distribution does S_n have? It will be useful to write S_n in terms of *standardizations* of L_j.

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A standardization of L_j (or of any random variable) is

$$\widetilde{L_j} = \frac{L_j - \mathbb{E}L_j}{\sqrt{\mathrm{Var} L_j}}, \quad \Rightarrow \mathbb{E}\widetilde{L}_j = 0, \quad \mathrm{Var}\,\widetilde{L}_j = 1$$

i.e., it is a centered version of L_j, inversely scaled by its standard deviation: standardizations of random variables are mean-0 and variance-1.

In terms of the Binomial tree parameters p_n, we have that,

$$\mathbb{E}L_j = \mu h_n, \qquad \mathrm{Var} L_j = 4p_n(1-p_n)\sigma^2 h_n.$$

Note that this agrees with our real-world CRR approximation for large n: $\mathrm{Var} L_j \sim \sigma^2 h_n$.

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With the standardization of the L_j variables, we have,

$$S_0 \exp\left(\sum_{j=1}^{n} L_j\right) = S_n = S_0 \exp\left(\sum_{j=1}^{n}\left(\mathbb{E}L_j + \widetilde{L}_j\sqrt{\mathrm{Var}L_j}\right)\right).$$

The distribution of S_n

$$S_n = S_0 \exp\left(\sum_{j=1}^{n} \left(\mathbb{E}L_j + \widetilde{L}_j \sqrt{\mathrm{Var} L_j}\right)\right).$$

This expression allows to us understand the large-n behavior of S_n.

$$\lim_{n\to\infty} \mathbb{E}L_j \text{ "=" } \mathbb{E}L_j = \mu h_n$$

$$\sum_{j=1}^{n} \mathbb{E}L_j = \sum_{j=1}^{n} \mu h_n = \mu h_n n = \mu T$$

$$\sqrt{\mathrm{Var} L_j} = \underbrace{\sqrt{4 p_n (1-p_n)}}_{\approx 1}\, \sigma \underbrace{\sqrt{h_n}}_{\sqrt{T}/\sqrt{n}} \Longrightarrow \sqrt{\mathrm{Var} L_j} \approx \sigma \sqrt{T} \frac{1}{\sqrt{n}}$$

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The goal is to take $n \uparrow \infty$.

Note that:

$$\lim_{n\uparrow\infty} \exp(\mu T) = \exp(\mu T), \qquad \lim_{n\uparrow\infty} \exp(\sigma\sqrt{T}\sqrt{4p_n(1-p_n)}) = \exp(\sigma\sqrt{T}).$$

But what about $\exp\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\widetilde{L}_j\right)$?

The central limit theorem

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- A direct corollary: the error of the empirical "Monte Carlo" mean scales like $\sqrt{\mathrm{Var} X_j}/\sqrt{n}$.

"MC" estimate of $\mathbb{E}X$ is $\frac{1}{n}\sum_{j=1}^{n} X_j$

How "big" is $\frac{1}{n}\sum_{j=1}^{n} X_j - \mathbb{E}X$?

$$= \frac{1}{n}\sum_{j=1}^{n} (X_j - \mathbb{E}X_j)$$

$$= \frac{\sqrt{\operatorname{Var} X}}{n}\sum_{j=1}^{n} \underbrace{\frac{X_j - \mathbb{E}X_j}{\sqrt{\operatorname{Var} X}}}_{\text{Standardized RV}}$$

$$= \sqrt{\frac{\operatorname{Var} X}{n}} \underbrace{\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \frac{X_j - \mathbb{E}X_j}{\sqrt{\operatorname{Var} X}}}_{\text{CLT: } N(0,1)}$$

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- A direct corollary: the error of the empirical "Monte Carlo" mean scales like $\sqrt{\mathrm{Var} X_j}/\sqrt{n}$.
- It is important that the X_j random variables *not* depend on ~~n~~. j

Back to the CRR model

We need to determine the n-asymptotic behavior of

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \widetilde{L}_j,$$

where the $\widetilde{L}_j$ are indeed iid, $\mathbb{E}\widetilde{L}_j = 0$, $\text{Var}\,\widetilde{L}_j = 1$

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The problem: The distribution of $\widetilde{L}_j$ *does* depend on ~~n~~. j.

To more formally understand why this is an issue: for each fixed n, we have the collection of random variables,

$$\widetilde{L}_{n,1}, \widetilde{L}_{n,2}, \ldots, \widetilde{L}_{n,n}, \quad \widetilde{L}_{n,j} = \frac{L_j - \mathbb{E}L_j}{\sqrt{\text{Var}L_j}} \qquad L_j \sim L_{n,j}$$

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The Central Limit Theorem as we've stated it does *not* directly tell us about the $n \to \infty$ limit of,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \widetilde{L}_{n,j}.$$

The fix

Although the distribution of $\widetilde{L}_{n,j}$ depends on n, for large n and m the distributions of $\widetilde{L}_{n,j}$ and $\widetilde{L}_{m,j}$ are actually quite similar.

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In our case, for example, we could write the $(n+1)$st summation as,

$$\frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} \widetilde{L}_{n+1,j} = \underbrace{\frac{1}{\sqrt{n+1}} \sum_{j=1}^{n} \widetilde{L}_{n,j} + \frac{1}{\sqrt{n+1}} \widetilde{L}_{n+1,n+1}}_{(a)} + \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n} \left[\widetilde{L}_{n+1,j} - \widetilde{L}_{n,j} \right].$$

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Term (a) is a sum of $n+1$ independent random variables scaled by $1/\sqrt{n+1}$, but the $(n+1)$st summand is *not* identically distributed.

Hence, if we had a Central Limit Theorem for *non*-identically distributed random variables, we could tackle this case.

The Lindeberg Condition

Let $\{X_j\}_{j=1}^{\infty}$ be independent and mean-zero, but not identically distributed.

Definition (Lindeberg's condition)

Let $\Sigma_n^2 := \sum_{j=1}^n \mathrm{Var} X_j$.

Lindeberg's condition is the following on the sequence $\{X_j\}_{j=1}^{\infty}$: For every $\epsilon > 0$, we have,

$$\lim_{n\uparrow\infty} \frac{\sum_{j=1}^n \mathbb{E}\left[X_j^2 \mathbb{1}_{|X_j| > \epsilon \Sigma_n}\right]}{\Sigma_n^2} = 0.$$

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Suppose $\{X_j\}_{j=1}^{\infty}$ are independent and mean-zero, and satisfy Lindeberg's condition. Then,

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The upshot for us: so long as our random variables satisfy the appropriate version of Lindeberg's condition, then we can use the Central Limit Theorem.

Lindeberg's condition

For our ("triangular") sequence of random variables, $\{\widetilde{L}_{n,j}\}_{j=1}^{n}$ with $n \in \mathbb{N}$, Lindeberg's condition for this setup is: For every $\epsilon > 0$,

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\left[\widetilde{L}_{n,j}^2 \mathbb{1}_{|\widetilde{L}_{n,j}|>\sqrt{n}\epsilon}\right] = 0.$$

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This holds in our particular case, which implies:

$$\lim_{n\uparrow\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \widetilde{L}_{n,j} \sim \mathcal{N}(0,1).$$

$$\frac{1}{\sqrt{n}} = \frac{1}{\Sigma_n} : \Sigma_n^2 = \sum_{j=1}^{n} \mathrm{Var}\, X_j = \sum 1 = n$$

Back to securities

Finally, recall that we started with the assertions:

$$S_n = S_0 \exp\left(\mu T + \sqrt{4p_n(1-p_n)}\sigma\sqrt{T}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\widetilde{L}_j\right)$$

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Therefore, if $X \sim \mathcal{N}(0,1)$, then

$$\lim_{n\uparrow\infty} S_n \sim S_0\exp(\mu T + X\sigma\sqrt{T}).$$

Put another way: if $Z \sim \mathcal{N}(\mu T, (\sigma\sqrt{T})^2)$, then

$$\lim_{n\uparrow\infty} S_n \sim S_0\exp(Z), = \exp(\log S_0 + Z)$$

i.e., the continuous-time limit of S_n is the exponential of a normally distributed random variable.

The continuous-time price

If we let $S(T)$ denote the $n \uparrow \infty$ limit of S_n, we conclude that,

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A random variable that is the exponential of a normal random variable is called a **lognormal** random variable.

I.e., our continuous-time security price is a lognormal random variable, which is typically written as,

$$S(T) \sim \text{lognormal}(\mu T + \log S_0, \sigma^2 T).$$

mean of the log of the variable

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– Note that T is arbitrary; e.g., the same rationale implies that $S(T/2)$ is also a lognormal random variable.
– It is not true that $\mathbb{E}S(T) = \mu T$ or $\mathbb{E}S(T) = \exp(\mu T)$. In fact, one can show that

$$\mathbb{E}S(T) = \exp\left(\mu T + \frac{\sigma^2}{2} T\right).$$

(HW problem)

Note that this matches our expression for the mean from last time.

Modeling continuous-time prices

For any $t > 0$, our continuous-time CRR model states:

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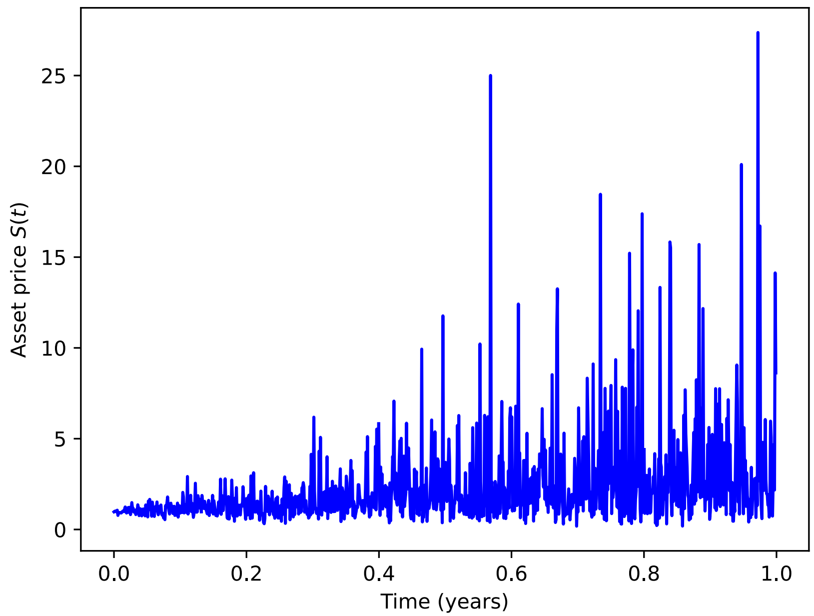
How would we simulate a trajectory given (S_0, μ, σ^2)?



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1. Generate $Z \sim \mathcal{N}(\mu t + \log S_0, \sigma^2 t)$
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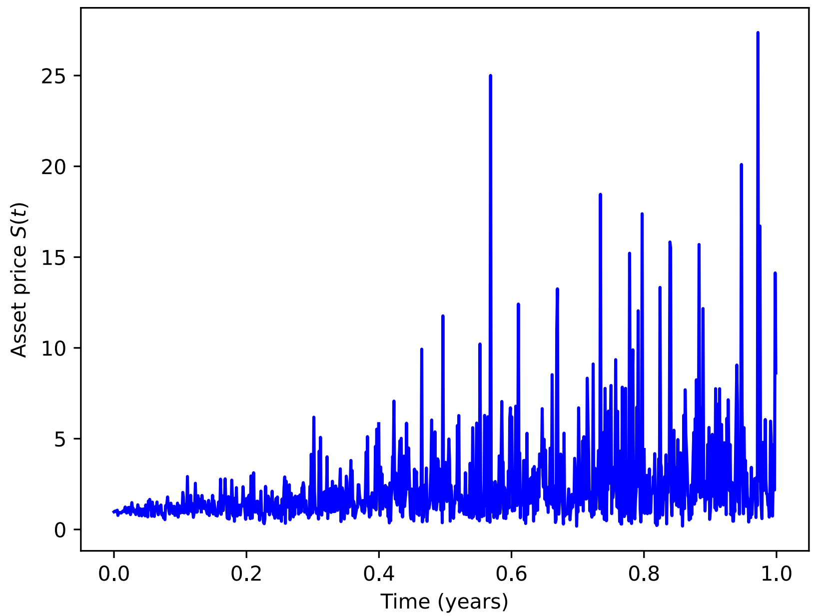
(It's true that above $S(t)$ as the correct distribution.)



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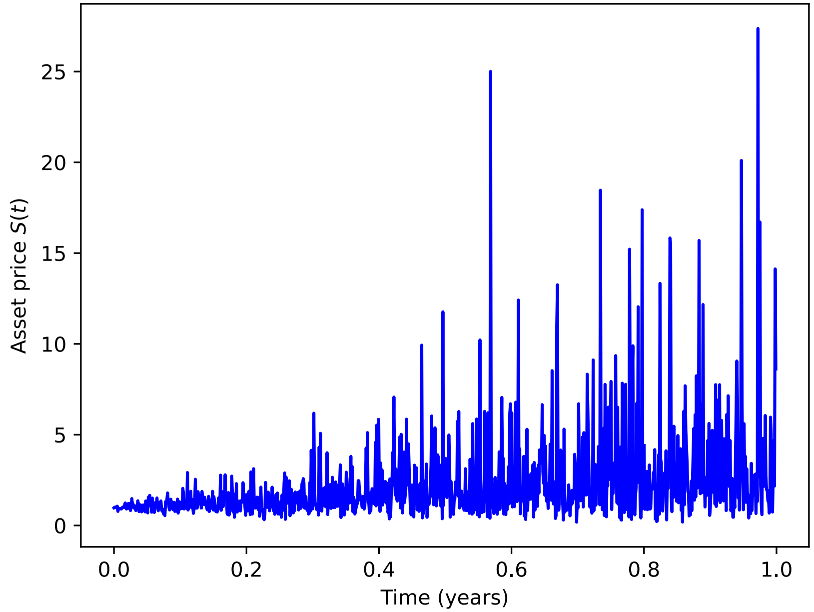
This, unfortunately, does not produce what we expect:



Temporal structure

The missing piece of the puzzle for us is the temporal structure of the signal: consider the model

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for very small $t \ll 1$.

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What is not accounted for is the Markovian structure of this process: I.e., while $S(T)$ has the distributed specified, if we are provided that the price at time $T - \epsilon$ is $S(T - \epsilon) = s$, then $S(T)$ *conditioned* on this value has a lognormal distribution with small parameters:

$$S(T) \,\big|\, S(T - \epsilon) \sim \text{lognormal}(\log S(T - \epsilon) + \mu\epsilon, \sigma^2 \epsilon)$$

Temporal structure

The missing piece of the puzzle for us is the temporal structure of the signal: consider the model

$$S(t) \sim \text{lognormal}(\mu t + \log S_0, \sigma^2 t).$$

for very small $t \ll 1$.

In this case, $S(t)$ is "very close" to $\exp \log S_0 = S_0$. This fact is reflected in the generated image.

What is not accounted for is the Markovian structure of this process: I.e., while $S(T)$ has the distributed specified, if we are provided that the price at time $T - \epsilon$ is $S(T - \epsilon) = s$, then $S(T)$ *conditioned* on this value has a lognormal distribution with small parameters:

$$S(T) \,\big|\, S(T - \epsilon) \sim \text{lognormal}(\log S(T - \epsilon) + \mu\epsilon, \sigma^2 \epsilon)$$

I.e., $S(T)$ should be constrained to lie "close" to $S(T - \epsilon)$, and in particular the asset price should be continuous in time.

We have not captured this structure by only inspecting the distribution.

To more formally understand these concepts, we'll need to introduce stochastic processes (next time).

Risk-neutrality redux

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Hence, in a risk-neutral "world", the values u_n and d_n should match their values in the marketplace, i.e., due to the real-world CRR equations:

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad \text{(risk neutral)}$$

The risk-neutral probability

What does change in a risk-neutral world is the probabilistic structure, i.e., p_n.

We choose p_n so that,

$$\mathrm{FV}(S(t_0)) = \mathbb{E}S(t_1).$$

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Recall that this is provided by

- the actual risk-free rate $r > 0$ (e.g., from risk-free securities)
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Hence: $\mathrm{FV}(S(t_0)) = e^{(r-q)h_n} S(t_0)$. Using this in the risk-neutrality condition, we have,

$t_1 = t_0 + h_n$

$$e^{(r-q)h_n} = p_n u_n + (1-p_n) d_n,$$

i.e.,

no-arbitrage market $\Longrightarrow d_n < e^{h_n(r-q)} < u_n$

$$p_n = \frac{e^{(r-q)h_n} - d_n}{u_n - d_n}. \in (0,1)$$

(Recall a convenient fact: assuming a no-arbitrage market implies $0 < p_n < 1$.)

The risk-neutral CRR model

Real-world CRR: $p_n = \frac{1}{2}\left(1 + \frac{\sqrt{h_n}}{\sigma}\mu\right)$

The risk-neutral CRR model has the conditions:

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad p_n = \frac{e^{(r-q)h_n} - d_n}{u_n - d_n}.$$

What about large n?

$$u_n = \exp(\sigma\sqrt{h_n}) \sim 1 + \sigma\sqrt{h_n} + \tfrac{1}{2}\sigma^2 h + \cdots \qquad (h_n \ll 1)$$

$$u_n - d_n \approx 2\sigma\sqrt{h_n} + \cdots$$

$$e^{(r-q)h_n} \sim 1 + (r-q)h_n + \cdots$$

$$\Rightarrow p_n \approx \frac{1 + (r-q)h_n - 1 + \sigma\sqrt{h_n} - \frac{\sigma^2}{2}h_n + \cdots}{2\sigma\sqrt{h_n} + \cdots} \approx \frac{1}{2} + \frac{(r-q)}{2\sigma}\sqrt{h_n} - \frac{\sigma}{4}\sqrt{h_n}$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{h_n}}{\sigma}\left(r - q - \frac{\sigma^2}{2}\right)\right)$$

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With some analysis (similar to the standard CRR model), one can determine that for large n, one has the valid approximationsk

$$u_n = \exp(\sigma_{RN}\sqrt{h_n}), \qquad d_n = \exp(-\sigma_{RN}\sqrt{h_n}), \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu_{RN}}{\sigma_{RN}}\sqrt{h_n}\right),$$

where (μ_{RN}, σ_{RN}) are the risk-neutral drift and volatility, which satisfy:

$$\sigma_{RN} = \sigma, \qquad \mu_{RN} = r - q - \frac{\sigma^2}{2}$$

Hence, one can use these equations to set (p_n, u_n, d_n) for a risk-neutral CRR tree.

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Hence, one can use these equations to set (p_n, u_n, d_n) for a risk-neutral CRR tree.

Note that under this model,

$$\mathbb{E}S(T) = S_0 \exp(\mu_{RN}T + T\sigma_{RN}^2/2) = S_0 \exp[(r-q)T]$$

precisely as expected from a risk-neutral model.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.