Math 5760/6890: Introduction to Mathematical Finance
Cox-Ross-Rubinstein model

See Petters and Dong 2016, Section 5.2

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The binomial tree pricing and CRR models

We have modeled a security's price $S_j = S(t_j)$ via,

$$S_{j+1} = G_{j+1} S_j, \qquad\qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

From this model, we've concluded:

- $L := \log(S_n/S_0)$ is a scaled/shifted $\mathrm{Binomial}(n,p)$ random variable.
- $S_n = S_0 e^L$ is the exponential of a scaled/shifted Binomial random variable
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The CRR model places the following additional constraints on our standard Binomial tree model:

- Geometric symmetry of tree prices: $u = 1/d$
- The continuous-time limit of the expected log-return matches the real-world drift:

$$\mu = \lim_{n\to\infty} \frac{1}{h_n} \mathbb{E} L_j$$

- The continuous-time limit of the variance of the log-return matches the real-world (squared) volatility:

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{h_n} \mathrm{Var} L_j$$

Hence, for finite n, (p, u, d) should depend on the time discretization parameter n. I.e.:

$$(p, u, d) = (p_n, u_n, d_n).$$

Goal: use CRR constraints to choose (p_n, u_n, d_n).

The problem setup

We seek to construct a fixed, finite-n Binomial tree model over the time period $[0, T]$. I.e., we seek to compute (p_n, u_n, d_n) for a fixed n and $h_n = T/n$.

We assume that the (continuous-time) drift and volatility parameters (μ, σ) are available to us.
(E.g., we've computed approximations to them from historical data.)

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- $d_n = 1/u_n \Longrightarrow$ need only determine (p_n, u_n).
- Drift matching: Ideally, we have,

$$\frac{\mathbb{E}L_j}{h_n} = \mu_n \xrightarrow{n\uparrow\infty} \mu.$$

  We can't really do this directly since we only want to construct (p_n, u_n) for finite n. Hence, we will instead impose:

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- We have a similar condition for matching the volatility:

$$\frac{\mathrm{Var}L_j}{h_n} = \sigma_n^2 \approx \sigma^2$$

The CRR setup, I

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$d_n = 1/u_n$

Of course, we know how to compute these: We have,

$$L_j = \log d_n + X_j \log \frac{u_n}{d_n} = -\log u_n + 2X_j \log u_n,$$

where $X_j \sim \mathrm{Bernoulli}(p_n)$.

Recall: $\mathbb{E} X_j = p_n \cdot 1 + (1-p_n) \cdot 0 = p_n$

$\mathrm{Var}\, X_j = \mathbb{E}(X_j - p_n)^2 = p_n (1-p_n)^2 + (1-p_n)(-p_n)^2$

$= p_n(1-p_n)[1-p_n+p_n] = p_n(1-p_n)$

$\Rightarrow \mathbb{E} L_j = -\log u_n + 2p_n \log u_n = (2p_n - 1)\log u_n$

$$\operatorname{Var} L_j = (2 \log u_n)^2 \operatorname{Var} X_j = 4 p_n (1-p_n)(\log u_n)^2$$

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Therefore:

$$\mathbb{E} L_j = (2p_n - 1) \log u_n, \qquad \mathrm{Var} L_j = 4 p_n (1 - p_n) (\log u_n)^2$$

The CRR setup, II

Therefore, our CRR constraints are the following:

$$d_n = \frac{1}{u_n}$$

$$\mu = \frac{2p-1}{h_n} \log u_n \Longrightarrow \mu^2 = \frac{(2p-1)^2}{h_n^2} (\log u_n)^2$$

$$\sigma^2 = \frac{4p_n(1-p_n)}{h_n} (\log u_n)^2.$$

where we have replaced some instances of "≈" with "=".

$$\Longrightarrow \quad \mu^2/\sigma^2 = \frac{1}{h_n}\left[\frac{(2p_n-1)^2}{4p_n(1-p_n)}\right] \quad : \quad \delta := h_n \mu^2/\sigma^2$$

$$\delta = \frac{(2p_n-1)^2}{4p_n(1-p_n)}$$

$$\delta\left[-4p_n^2+4p_n\right] = 4p_n^2 - 4p_n + 1$$

$$0 = p_n^2\left[4+4\delta\right] + p_n\left[-4-4\delta\right] + 1$$

$$= 4(1+\delta)\,p_n^2 - 4(1+\delta)\,p_n + 1$$

$$0 = p_n^2 - p_n + \frac{1}{4(1+\delta)}$$

$$p_n = \frac{1}{2}\left[1 \pm \sqrt{1^2 - \frac{1}{1+\delta}}\right]$$

$$= \frac{1}{2}\left[1 \pm \sqrt{\frac{\delta}{1+\delta}}\right]$$

$$= \frac{1}{2}\left[1 \pm \frac{\sqrt{\delta}}{\sqrt{1+\delta}}\right]$$

Do we choose + or − above?

Recall: $\mu = \frac{(2p-1)}{h_n} \log u_n$

If $p_n = \frac{1}{2}\left(1 - \frac{\sqrt{\delta}}{\sqrt{1+\delta}}\right) < \frac{1}{2} \implies 2p_n - 1 < 0 \implies \mu < 0.$

unphysical restriction

$\implies$ choose "+" option: $p_n = \frac{1}{2}\left(1 + \frac{\sqrt{\delta}}{\sqrt{1+\delta}}\right)$

$$\delta = h_n \mu^2 / \sigma^2 \Rightarrow \text{for large } n, \quad \delta \ll 1$$

Then for large n: $\frac{1}{\sqrt{1+\delta}} \simeq 1$ ($f(x) = \frac{1}{\sqrt{1+x}}$, Taylor expand around $x=0$: $f(x) = 1 + \ldots$)

So: $p_n = \frac{1}{2}\left(1 + \frac{\sqrt{\delta}}{\sqrt{1+\delta}}\right) \simeq \frac{1}{2}\left(1 + \sqrt{\delta}\right)$

$$= \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right)$$

What about u_n?

$$\sigma^2 = \frac{4p_n(1-p_n)}{h_n}\left(\log u_n\right)^2$$

note: $p_n \simeq \frac{1}{2} + c \cdot \sqrt{h_n}$

large n: $p_n \to 1/2$

$\Downarrow$ $p_n \approx 1/2$

$$\sigma^2 = \frac{1}{h_n}\left(\log u_n\right)^2 \Rightarrow u_n = \exp\left(\sigma\sqrt{h_n}\right)$$

$$d_n = \frac{1}{u_n} = \exp\left(-\sigma\sqrt{h_n}\right)$$

The CRR setup, II

Therefore, our CRR constraints are the following:

$$\begin{aligned} d_n &= \frac{1}{u_n} \\ \mu &= \frac{2p-1}{h_n} \log u_n \\ \sigma^2 &= \frac{4p_n(1-p_n)}{h_n} (\log u_n)^2. \end{aligned}$$

where we have replaced some instances of "$\approx$" with "$=$".

After some computations and approximations, we arrive at the following *real-world CRR equations*:

$$p_n \approx \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right), \qquad u_n \approx \exp(\sigma\sqrt{h_n}), \qquad d_n \approx \exp(-\sigma\sqrt{h_n}).$$

The real-world CRR tree

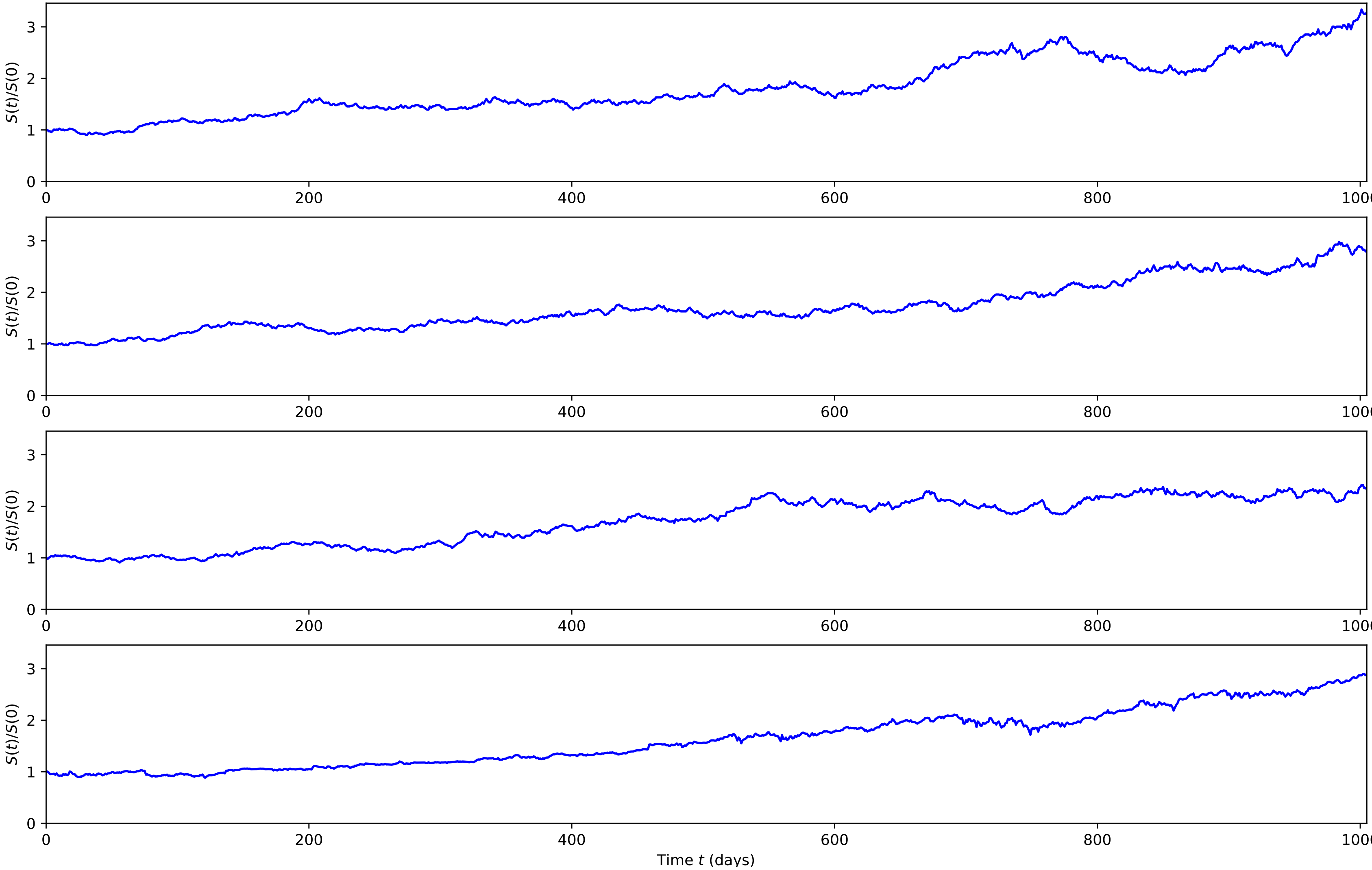
A "real-world" CRR tree/model is therefore constructed in the following way:

- Historical data is used to compute an asset's continuous-time drift and volatility (μ, σ)
- The terminal time T and number of periods n is determined. $h_n = T/n$.
- The real-world CRR equations are used to set (p_n, u_n, d_n):

$$p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right), \qquad u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}).$$

Let's play a game

Which is the simulated price?



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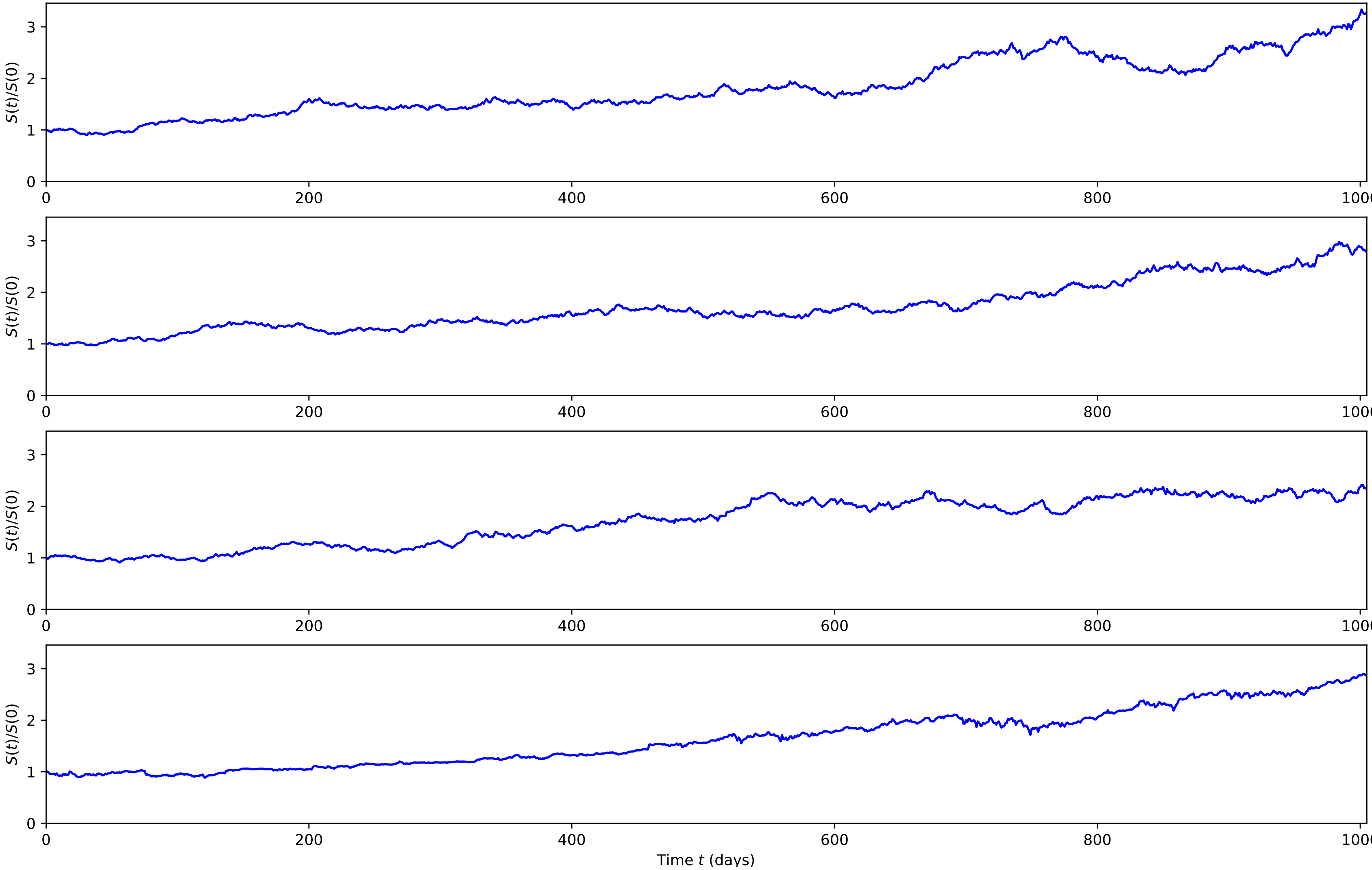
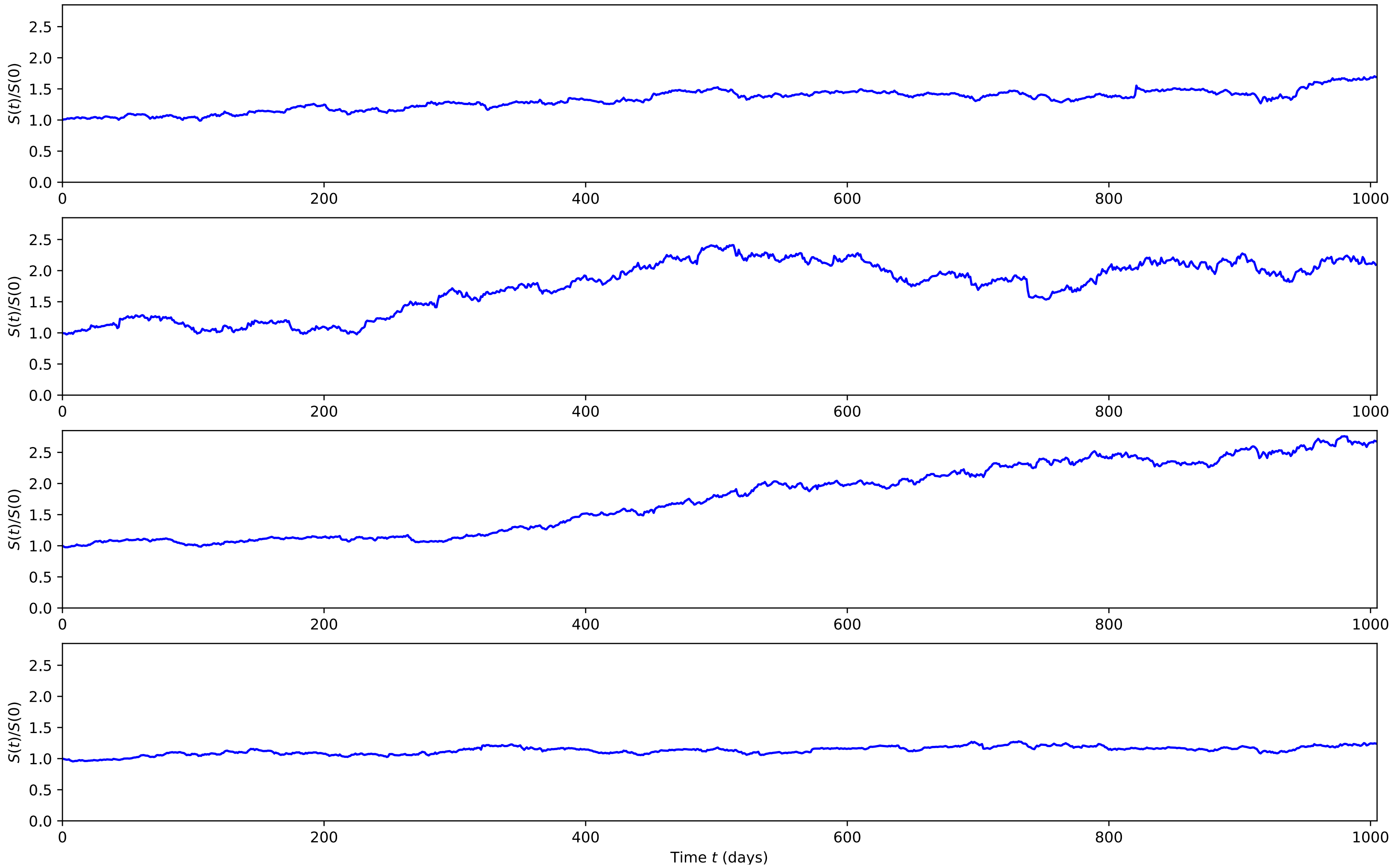


MSFT

Let's play a game, II

Which prices are simulated?



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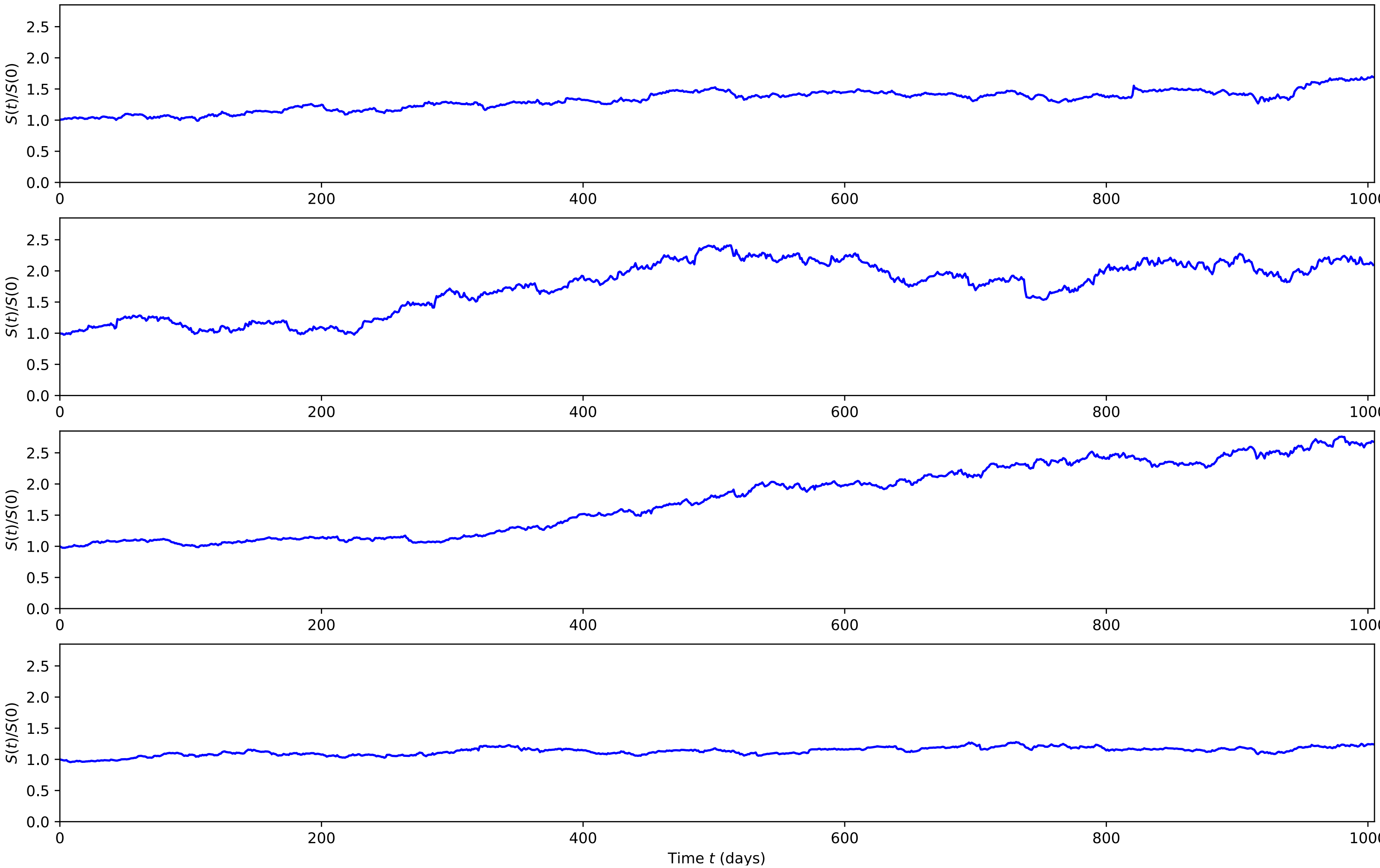

GE

Airbus

Lockheed Martin

Coca cola

Some CRR properties, I

Some initial observations about the tuple (p_n, u_n, d_n) of the real-world CRR model:

- Because $u_n = \exp(\sigma\sqrt{h_n})$, and $h_n = T/n$, then

$$\lim_{n\uparrow\infty} u_n = 1,$$

  and similarly for $d_n = 1/u_n$.
  I.e., the uptick and downtick geometric rates become very close to unity for large n.

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$$\lim_{n\uparrow\infty} u_n = 1,$$

  and similarly for $d_n = 1/u_n$.
  I.e., the uptick and downtick geometric rates become very close to unity for large n.
- Because $p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right)$, then

$$\lim_{n\uparrow\infty} p_n = \frac{1}{2},$$

  so that for large n the CRR tree tends toward fair coin flips.

Some CRR properties, II

What kind of statistics does S_n have under this model? We have,

$$S_n = S_0 e^L = S_0 \exp\left(\sum_{j=1}^{n} L_j\right).$$

We've seen that

$$\mathbb{E} S_n = S_0 \left(p_n u_n + (1 - p_n) d_n\right)^n ,$$

and we have the real-world CRR equations:

$$p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right), \qquad u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}).$$

$$(p_n u_n + (1-p_n) d_n)^n = \left[\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right) \exp(\sigma\sqrt{h_n}) + \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{h_n}\right) \exp(-\sigma\sqrt{h_n}) \right]^n$$

$$= \Big[\tfrac{1}{2}\left(\exp(\sigma\sqrt{h_n}) + \exp(-\sigma\sqrt{h_n})\right)$$
$$+ \tfrac{1}{2}\tfrac{\mu}{\sigma}\sqrt{h_n}\left(\exp(\sigma\sqrt{h_n}) - \exp(-\sigma\sqrt{h_n})\right)\Big]^n$$

$$= \Big[\cosh(\sigma\sqrt{h_n}) + \frac{\mu\sqrt{h_n}}{\sigma}\sinh(\sigma\sqrt{h_n})\Big]^n$$

recall: $h_n \ll 1$ $(n \gg 1)$

$\frac{d}{dx}\cosh(x) = \sinh(x)$ $\qquad \cosh(0) = 1$

$\frac{d}{dx}\sinh(x) = \cosh(x)$ $\qquad \sinh(0) = 0$

Taylor expansion:

$$f(x) = \cosh(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$$
$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$g(x) = \sinh(x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!} x^j$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$[p_n u_n + (1-p_n) d_n]^n = \left[\cosh(\sigma\sqrt{h_n}) + \frac{\mu\sqrt{h_n}}{\sigma}\sinh(\sigma\sqrt{h_n})\right]^n$$

$$= \left[1 + \frac{\sigma^2 h_n}{2} + \dots + \frac{\mu\sqrt{h_n}}{\sigma}\left(\sigma\sqrt{h_n} + \dots\right)\right]^n$$

$$\overset{h_n^2 \ll h_n}{\approx} \left[1 + \frac{\sigma^2}{2} h_n + \mu h_n\right]^n \qquad h_n = T/n$$

$$= \left[1 + \frac{1}{n}\left[\left(\mu + \frac{\sigma^2}{2}\right)T\right]\right]^n$$

$$\overset{n\uparrow\infty}{=} \exp\left(\left(\mu + \sigma^2/2\right)T\right)$$

$$\Rightarrow \mathbb{E} S_n = S_0 \exp\left(\left(\mu + \sigma^2/2\right)T\right) \quad (n \nearrow \infty)$$

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and we have the real-world CRR equations:

$$p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right), \qquad u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}).$$

These allow us to conclude:

$$\lim_{n\uparrow\infty} \mathbb{E} S_n = S_0 \exp\left[\left(\mu + \frac{\sigma^2}{2}\right) T\right],$$

i.e., there is a well-defined limit independent of the discretization parameters n and h_n.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.