# Math 5760/6890: Introduction to Mathematical Finance
# Binomial Options Pricing

See Petters and Dong 2016, Section 5.1

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# The tree pricing model



We've seen the basic anatomy of the binomial pricing model: Given $(p, u, d)$, then,

$S_{j+1} = G_{j+1} S_j$

$$S_j = G_j S_{j+1}, \qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

where $S_j = S(t_j)$ and $G_j$ is the gross return rate.

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where $S_j = S(t_j)$ and $G_j$ is the gross return rate. In this model, it turns out that *log returns* are particularly convenient to work with:

$$\frac{S_n}{S_0} = e^L, \qquad L = \sum_{j=1}^{n} L_j, \qquad L_j = \left\{ \begin{array}{ll} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{array} \right.$$

# Our $(p, u, d)$ assumptions

Recall that we've always assumed,

- $p \in (0, 1)$
- $d < 1 < u$

The $p \in (0, 1)$ assumption is reasonable: if $p = 0, 1$, then the model is not random, implying that there is no uncertainty about the future.

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The assumption that $u > d$ is just for convenience: if $u < d$, then consider another triple $(q, \tilde{u}, \tilde{d})$ with,

- $q = 1 - p$
- $\tilde{u} = d$
- $\tilde{d} = u < \tilde{u}$

Then the $(p, u, d)$ is equivalent in distribution to the $(q, \tilde{u}, \tilde{d})$ model, but the latter satisfies $\tilde{u} > \tilde{d}$.

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Hence, we assume $u > d$ without loss of generality.

Finally, the fact that 1 must be sandwiched between $d$ and $u$ is a requirement to ensure a no-arbitrage setup:

- If $d \geqslant 1$, then $\Pr(S_1 \geqslant S_0) = 1$, and $\Pr(S_1 > S_0) = p > 0$, ensuring an arbitrage by holding a long position in $S$.
- If $u \leqslant 1$, then $\Pr(S_1 \leqslant S_0) = 1$, and $\Pr(S_1 < S_0) = p > 0$, ensuring an arbitrage by holding a short position in $S$.

# Risk neutrality



There is one more concept that will be useful for us to employ in modeling investors:

Suppose I seek to sell you an asset today at a price $S_0$.

If you have a probabilistic model for the future trajectory of $S(t)$, and if your model predicts $\mathbb{E}S(t) < S_0$, then you have limited incentive purchase this asset.

On the other hand, if your model (correctly!) predicts $\mathbb{E}S(t) > S_0$, then there is opportunity for arbitrage, assuming you have unlimited capital to invest.
In such a case, we assume that another saavy investor would have already recognized this and removed the arbitrage opportunity through exploitation; hence your model is unlikely to be accurate.

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Based on these scenarios, then a reasonable assumption on a valid model is that $\mathbb{E}S(t) = S_0$.

A probabilistic model satisfying this assumption is said to be **risk neutral**.

More formally, a risk-neutral probability measure satisfies the above assumption.

# Risk neutrality in practice



Note that for our tree model, we have

$$\mathbb{E}S_1 = S_0 \mathbb{E}G_1 = S_0 \left( pu + (1-p)d \right),$$

The risk-neutrality requirement is that $\mathbb{E}S_1 = S_0$.

$$S_0(pu + (1-p)d) = S_0$$

$$p(u-d) = 1-d$$

$$p = \frac{1-d}{u-d} \implies \mathbb{E}S_1 = S_0$$

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Hence, in order for this single-period jump to be risk-neutral, then we require,

$$p = \frac{1-d}{u-d}.$$

This prescribes $p$ in terms of the single-period upward/downward factors.
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This one-period result extends to multiple periods through induction:

$$\mathbb{E}[S_{n+1} \,|\, S_n] = S_n \mathbb{E} G_n = S_n (pu + (1-p)d).$$

And again if $p$ is as given by the formula above, then $\mathbb{E} S_{n+1} = \mathbb{E} S_n = S_0$.

# Some real-world considerations, I

Such a model has limitations – e.g., in practice one would not be interested in the security if the average return was 0.

In practice, we assume an average return rate $m$ (in units matching those of $t$).

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Hence, assuming compounding corresponding to the number of periods in the model, then the present value of $S_1$ is,

$$\mathrm{PV}(S_1) = S_1 \left(1 + \frac{Tm}{n}\right) \overset{\Delta t = T/n}{=} S_1 \left(1 + m\Delta t\right),$$

so that the risk-neutral value of $p$ in this case is,

$$p = \frac{(1 + m\Delta t) - d}{u - d}$$

# Some real-world considerations, II



Another practicality worth building in: typically the asset $S$ is a stock.

Many stocks pay regular dividends, corresponding to a rate $q$.

A stock that pays dividends at rate $q$ should be discounted accordingly:

$$\mathrm{PV}(S_1) = S_1 \left(1 + \frac{Tr}{n} - \frac{Tq}{n}\right) \stackrel{\Delta t = T/n}{=} S_1 \left(1 + (m-q)\Delta t\right),$$

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The risk-neutral value of $p$ in this case is,

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Of course one expects that $m > q$ in order for the stock to be attractive to investors.

Note also that we must have

$$\Delta t < \frac{u}{m-q},$$

in order for this to be a valid model ($p < 1$).

# Options pricing



Although we have not really discussed options too much, one of the main applications of this model is in the pricing of options.

For simplicity, let's consider a ("European") call option:

- At $t = 0$, we are buying the right (not the requirement) to purchase a (say single) share of $S$. ("call")
- We can exercise this option *only* at time $t = T$ to purchase the stock at strike price $K$. ("European")

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Here is the basic logic of the options pricing model:

- We'll generate a probabilistic model for all time-$T$ outcomes of the options price.
- We'll propagate these prices *backward* in time through a pricing model.
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We'll assume that the following information is available:

- stock volatility (typically through historical data)
- a risk-free rate $r$, and a dividend rate $q$

# Options pricing, step 1: terminal time prices



Recall that the period-$n$ outcomes of the binomial model are:

$$S_0 u^n d^0, \;\; S_0 u^{n-1} d^1 \;\; , \ldots, \;\; S_0 u^1 d^{n-1}, \;\; S_0 u^0 d^n.$$

We'll *assume* that $(u, d)$ are given and prescribed.
(We'll soon see that a common model is to assign $(u, d)$ based on the historic volatility of the stock.)

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Note that the time-$T$ value of the call option is the value of the call relative to the strike price:

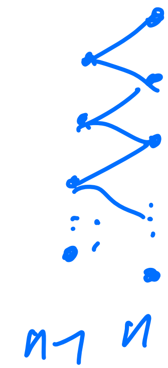
$$\widehat{S}_{n,j} := \max\{0, K - \widetilde{S}_{n,j}\}$$

I.e., the stock has value $K - \widetilde{S}_{n,j}$ should this difference be positive, but is 0 otherwise as we simply choose not to exercise the option.

# Options pricing, step 2: back-propagation under a risk-neutral measure

We know values at period $n$. How might be propoagate these prices back to time $n-1$?

There are $n$ values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that we must determine.



Using the binomial tree, we use risk neutrality:

- We define a risk-neutral measure by identifying $p$ appropriately:

$$p = \frac{(1 + (m - q)\Delta t) - d}{u - d}$$

- The value $\widehat{S}_{n-1,j}$ should be the expected value of the time-$n$ security under the risk-neutral measure:

$$\widetilde{S}_{n-1,j} = p\widehat{S}_{n,j} + (1-p)\widehat{S}_{n,j+1}, \qquad j = 0, \ldots, n$$

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This results in values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that are modeled prices of the option value at time $n-1$.

# Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

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An extra detail: different types of options have different rules of exercise.

In American options, one can exercise the option at any time up until time $T$.

Under an American option, the risk-neutral expectation should be modified not only by a present value discount, but also by the possibility of exercise:

- At period $k$, compute the standard binomial tree value of the stock at the current period (call this say $S_{k,j}$). $S_{k,j} = S_0 u^j d^{k-j}$

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The idea is that if exercise is possible at a certain time, we should model it.

# References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.