Math 5760/6890: Introduction to Mathematical Finance
Binomi~~d~~al Pricing Models

See Petters and Dong 2016, Section 5.1

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

October 24, 2023

www.sci.utah.edu

The tree pricing model

The overall goal:

> Given a security's time-0 price $S(0)$, construct a probabilistic model for $S(t)$, $t > 0$.

We do this with discrete time steps:

- Divide $[0, T]$ into $n \in \mathbb{N}$ equally sized intervals, $[t_j, t_{j+1}]$ for $j = 0, \ldots, N$.
- $t_j = jh$ with $h = T/n$
- Let $S_j = S(t_j)$
- Model $S_j \mapsto S_{j+1}$ as a multiplicative process:

$$S_{j+1} = G_{j+1} S_j = \begin{cases} u S_n, & \text{with probability } p \\ d S_n, & \text{with probability } 1 - p \end{cases}$$

- We assume (p, u, d) satisfies $p \in (0, 1)$, and $d < 1 < u$.

The tree pricing model

The overall goal:

Given a security's time-0 price $S(0)$, construct a probabilistic model for $S(t)$, $t > 0$.

We do this with discrete time steps:

- Divide $[0, T]$ into $n \in \mathbb{N}$ equally sized intervals, $[t_j, t_{j+1}]$ for $j = 0, \ldots, N$.
- $t_j = jh$ with $h = T/n$
- Let $S_j = S(t_j)$
- Model $S_j \mapsto S_{j+1}$ as a multiplicative process:

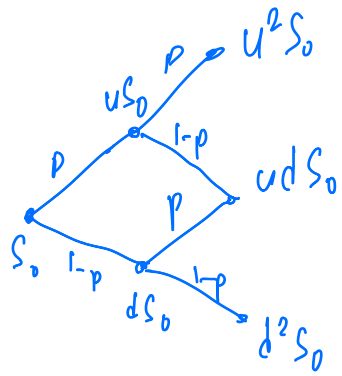
$$S_{j+1} = G_{j+1} S_j = \begin{cases} uS_n, & \text{with probability } p \\ dS_n, & \text{with probability } 1-p \end{cases}$$

- We assume (p, u, d) satisfies $p \in (0, 1)$, and $d < 1 < u$.
- The behavior over the entire period corresponds to an accumulation of multiplicative gross returns G_j:

$\log = \log_e$

$$S_n = S_0 \prod_{j=1}^{n} G_j, \qquad L := \log \frac{S_n}{S_0} = \sum_{j=1}^{n} \log G_j = \sum_{j=1}^{n} L_j$$

- The random variables $\{G_j\}_{j=1}^n$ are iid (shifted) Bernoulli random variables (Same for $\{L_j\}_{j=1}^n$.)
- Today: we'll describe more details about this model.



$u^2 S_0$
p
uS_0
$1-p$
p
udS_0
p
S_0
$1-p$
dS_0
$1-p$
$d^2 S_0$

A digression: Binomial random variables

Let $X \sim \mathrm{Bernoulli}(p)$, and consider n iid copies of X, $\{X_j\}_{j=1}^n$.

The random variable X satisfies,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

X is the outcome of a (binary) coin toss, where the outcome is biased if $p \neq 1/2$.

A digression: Binomial random variables

Let $X \sim \mathrm{Bernoulli}(p)$, and consider n iid copies of X, $\{X_j\}_{j=1}^n$.

The random variable X satisfies,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

X is the outcome of a (binary) coin toss, where the outcome is biased if $p \neq 1/2$. Of particular interest in several areas of probability and statistics is the random variable,

$$Y = \sum_{j=1}^n X_j,$$

which counts the number of $+1$ outcomes (say "heads" outcomes) of the X_j.

A digression: Binomial random variables

Let $X \sim \text{Bernoulli}(p)$, and consider n iid copies of X, $\{X_j\}_{j=1}^n$.

The random variable X satisfies,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

X is the outcome of a (binary) coin toss, where the outcome is biased if $p \neq 1/2$. Of particular interest in several areas of probability and statistics is the random variable,

$$Y = \sum_{j=1}^{n} X_j,$$

which counts the number of $+1$ outcomes (say "heads" outcomes) of the X_j.

Evidently, Y is a discrete random variable, with outcomes $\{0, 1, \ldots, n\}$.
Y is called a **Binomial**(n, p) random variable, and it has a Binomial distribution.

What is the probability of the outcome $\underbrace{HHH \cdots H}_{k}\,\underbrace{TT \cdots T}_{n-k}$

$\text{Prob} = p^k (1-p)^{n-k}$

A digression: Binomial random variables

Let $X \sim \mathrm{Bernoulli}(p)$, and consider n iid copies of X, $\{X_j\}_{j=1}^n$.

The random variable X satisfies,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

X is the outcome of a (binary) coin toss, where the outcome is biased if $p \neq 1/2$. Of particular interest in several areas of probability and statistics is the random variable,

$$Y = \sum_{j=1}^n X_j,$$

which counts the number of $+1$ outcomes (say "heads" outcomes) of the X_j.

Evidently, Y is a discrete random variable, with outcomes $\{0, 1, \ldots, n\}$.
Y is called a **Binomial**(n, p) random variable, and it has a Binomial distribution.

Some simple first- and second-order statistics are immediately computable:

$$\mathbb{E}Y = \sum_{j=1}^n \mathbb{E}X_j = \sum_{j=1}^n p = np, \qquad \mathrm{Var}Y \overset{(*)}{=} \sum_{j=1}^n \overset{p(1-p)}{\mathrm{Var}X_j} = np(1-p)$$

where $(*)$ uses the fact that the variance of the sum of independent random variables is the sum of their variances.

The Binomial distribution

The distribution (mass function) of Y is,

$$p_Y(k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}, \qquad 0 \leqslant k \leqslant n$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!\,(n-k)!}, \quad 0! = 1$$

The Binomial distribution

The distribution (mass function) of Y is,

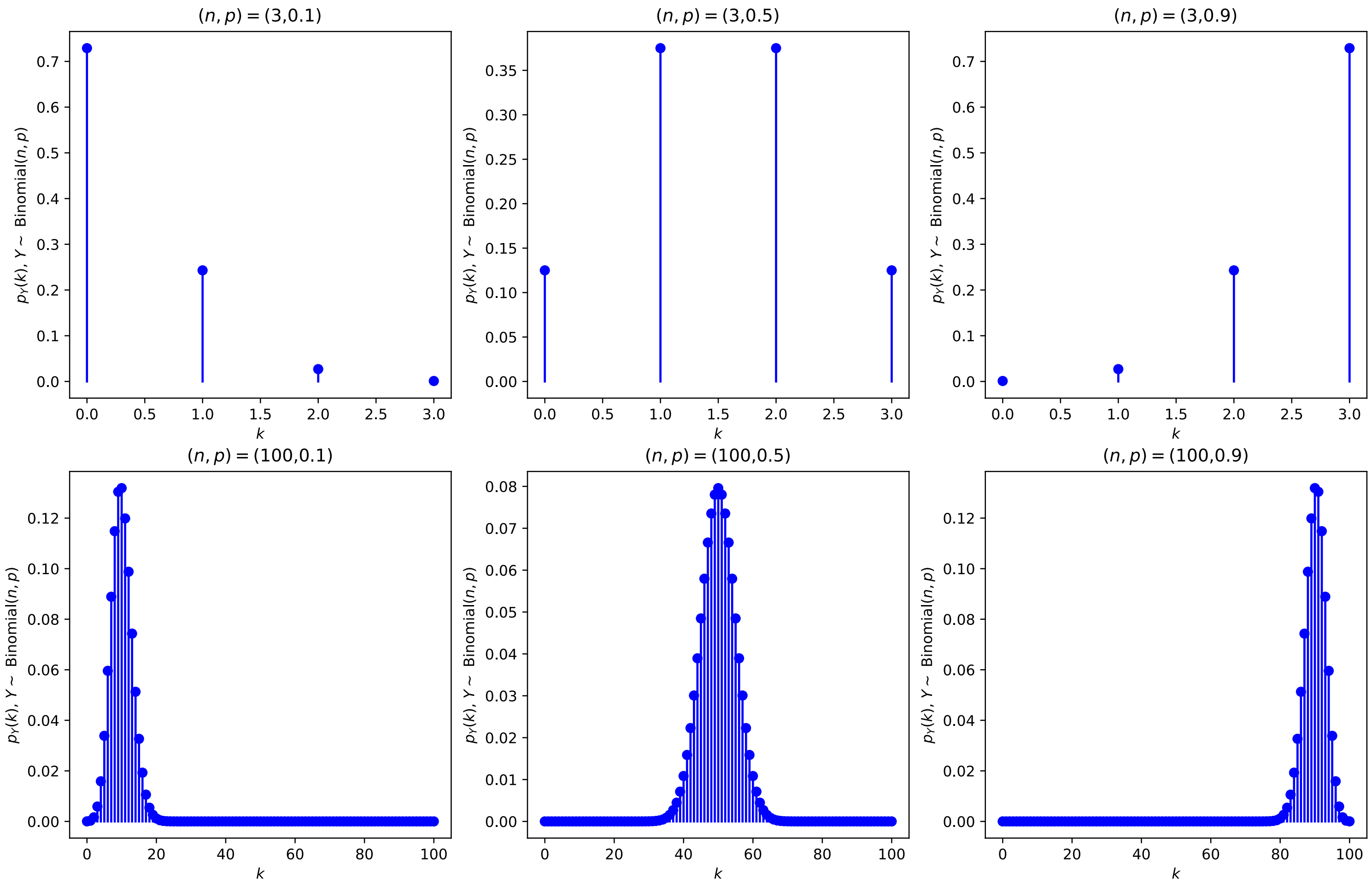
$$p_Y(k) = \left(\begin{array}{c} n \\ k \end{array}\right) p^k (1-p)^{n-k}, \qquad\qquad 0 \leqslant k \leqslant n$$

This is a valid mass function due to the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k y^{n-k} \quad \Longrightarrow \quad \sum_{k=0}^{n} p_Y(k) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) p^k (1-p)^{n-k}$$
$$= (p + (1-p))^n = 1.$$

If Y $\sim$Binomial(n,p), the quantity $p_Y(k)$ is the probability that we observe exactly k "heads" outcomes from n independent coin flips biased to land on heads with probability p.

The Binomial distribution mass function



$\mathbb{E}\, Y = np$

Shifting the Binomial distribution

The n Bernoulli trials need not have outcome $+1$ or 0.

Suppose that the Bernoulli-type random variable $\widetilde{X}$ has distribution,

$$\widetilde{X} = \begin{cases} b, & \text{with probability } p \\ a, & \text{with probability } 1-p \end{cases}$$

for any numbers a, b. Then note that if we write

$$\widetilde{X} = (b-a)X + a,$$

then $X \sim \text{Bernoulli}(p)$.

Shifting the Binomial distribution

The n Bernoulli trials need not have outcome $+1$ or 0.

Suppose that the Bernoulli-type random variable $\widetilde{X}$ has distribution,

$$\widetilde{X} = \begin{cases} b, & \text{with probability } p \\ a, & \text{with probability } 1-p \end{cases}$$

for any numbers a, b. Then note that if we write

$$\widetilde{X} = (b-a)X + a,$$

then $X \sim \text{Bernoulli}(p)$.

Then if we consider a sum of n iid copies of $\widetilde{X}$:

$$\widetilde{Y} := \sum_{j=1}^{n} \widetilde{X}_j, \qquad\qquad \widetilde{X}_j \overset{\text{iid}}{\sim} \widetilde{X},$$

then we observe that,

$$\widetilde{Y} = na + (b-a)Y,$$

where Y ~Binomial(n, p) is a ("standard") Binomial random variable.

Shifting the Binomial distribution

The n Bernoulli trials need not have outcome $+1$ or 0.

Suppose that the Bernoulli-type random variable $\widetilde{X}$ has distribution,

$$\widetilde{X} = \begin{cases} b, & \text{with probability } p \\ a, & \text{with probability } 1-p \end{cases}$$

for any numbers a, b. Then note that if we write

$$\widetilde{X} = (b-a)X + a,$$

then $X \sim \text{Bernoulli}(p)$.

Then if we consider a sum of n iid copies of $\widetilde{X}$:

$$\widetilde{Y} := \sum_{j=1}^{n} \widetilde{X}_j, \qquad \widetilde{X}_j \overset{\text{iid}}{\sim} X,$$

then we observe that,

$$\widetilde{Y} = na + (b-a)Y,$$

where $Y \sim$Binomial(n,p) is a ("standard") Binomial random variable.

Hence: iid sums of scaled/shifted Bernoulli random variables are just scaled/shifted Binomial random variables.

Back to pricing

Recall that the log-return in our security model satisfies,

$$L = \sum_{j=1}^{n} L_j,$$

with L_j iid, having distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Note: $\dfrac{L_j - \log d}{\log(u/d)}$ is Bernoulli(p)

Back to pricing

Recall that the log-return in our security model satisfies,

$$L = \sum_{j=1}^{n} L_j,$$

with L_j iid, having distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Hence, we essentially know everything about L:

- L_j is a shifted Bernoulli(p) random variable. Precisely, $L_j = \log d + X_j \log \frac{u}{d}$, where $X_j \sim \text{Bernoulli}(p)$.
- L is a shifted Binomial(n, p) random variable: $L = n \log d + Y \log \frac{u}{d}$, where $Y \sim \text{Binomial}(n, p)$.

(This is essentially why this is called the Binomial pricing model.)

Back to pricing

Recall that the log-return in our security model satisfies,

$$L = \sum_{j=1}^{n} L_j,$$

with L_j iid, having distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Hence, we essentially know everything about L:

- L_j is a shifted Bernoulli(p) random variable. Precisely, $L_j = \log d + X_j \log \frac{u}{d}$, where $X_j \sim \text{Bernoulli}(p)$.
- L is a shifted Binomial(n, p) random variable: $L = n \log d + Y \log \frac{u}{d}$, where $Y \sim \text{Binomial}(n, p)$.

(This is essentially why this is called the Binomial pricing model.)

Some caveats: S_n is not a Binomial random variable. Recall that $S_n = S_0 \exp L$.
The problem is that e^L is not Binomial (although its outcomes have the same probabilities as a Binomial random variable, they are not equispaced outcomes).

Some examples

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

– Compute the mean and variance of the terminal-time log-return.

$$L = \sum_{j=1}^{10} L_j \qquad L = 10 \cdot \log d + \log\left(\frac{u}{d}\right) Y, \qquad Y \sim \text{Binomial}(10, 0.6)$$

$$\mathbb{E} L = 10 \log d + \log\left(\frac{u}{d}\right) \underbrace{\mathbb{E} Y}_{np = 10 \cdot 0.6 = 6}$$

$$= 10 \log 0.9 + 6 \log\left(\frac{1.1}{0.9}\right)$$

$$\operatorname{Var} L = \operatorname{Var}\left(10 \log d + \log\left(\frac{u}{d}\right) Y\right) = \left(\log(u/d)\right)^2 \overbrace{\operatorname{Var} Y}^{np(1-p) = 6 \cdot 0.4 = 2.4} = 2.4 \left(\log \frac{1.1}{0.9}\right)^2$$

Some examples

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?

$$\frac{S_{10}}{S_0} = \prod_{j=1}^{10} G_j \implies \mathbb{E} S_{10} = S_0 \, \mathbb{E} \prod_{j=1}^{10} G_j$$

$$S_j = G_j S_{j-1}$$

Recall: if U, V are independent RV's, then

$$\mathbb{E}(UV) = (\mathbb{E}U)(\mathbb{E}V)$$

$$\implies \mathbb{E} \prod_{j=1}^{10} G_j = \prod_{j=1}^{10} \mathbb{E} G_j = \prod_{j=1}^{10} (pu + (1-p)d) = (0.66 + 0.36)^{10}$$

$$= (1.02)^{10} \implies \mathbb{E} S_{10} = S_0 (1.02)^{10}$$

Some examples

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?

Share price outcomes:

$d^{10} S_0, \; d^9 u S_0, \; d^8 u^2 S_0 \; \cdots \; , \; d^1 u^9 S_0, \; u^{10} S_0$

$\Rightarrow$ for which k values is $u^k d^{10-k} \geq 1$?

$k=5: \; u^5 d^5 \approx 0.95, \quad k=6: \; u^6 d^4 = 1.16$

$\Longrightarrow \quad u^k d^{10-k} \geq 1 \quad \text{when} \quad k \geq 6$

$$\Pr\left(\prod_{j=1}^{10} G_j \geq 1\right) = \Pr(Y \geq 6) \quad \text{where } Y \sim \text{Binomial}(10, 0.6)$$

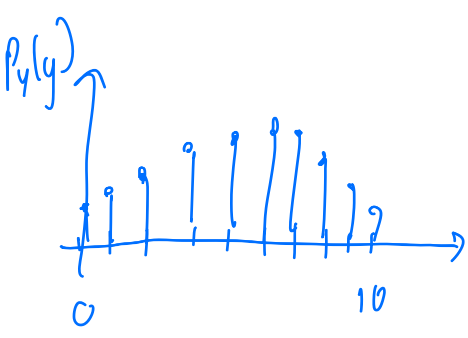
$$= \sum_{k=6}^{10} \binom{10}{k} p^k (1-p)^{10-k}$$

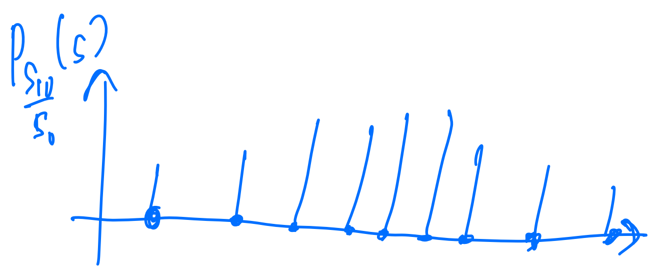
$$= \sum_{k=6}^{10} \binom{10}{k} 0.6^k \, 0.4^{10-k}$$

$$= \binom{10}{6} 0.6^6 \, 0.4^4 + \binom{10}{7} 0.6^7 \, 0.4^3 + \binom{10}{8} 0.6^8 \, 0.4^2$$

$$+ \binom{10}{9} 0.6^9 \, 0.4^1 + \binom{10}{10} 0.6^{10} \quad > 1/2 \; ? \quad = 1/2 \quad < 1/2$$

$$G := \prod_{j=1}^{10} G_j = \begin{cases} u^{10} & \text{w/ prob } p^{10}\binom{10}{10} \leftarrow \Pr(Y=10) \\ u^9 d & \text{w/ prob } p^9(1-p)\binom{10}{9} \leftarrow \Pr(Y=9) \\ u^8 d^2 & \text{w/ prob } p^8(1-p)^2\binom{10}{8} \leftarrow \Pr(Y=8) \\ \vdots & \end{cases}$$





Some examples

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?
- What is the probability that $S_5 \geqslant S_0$?

Same as before, but $n=5$

for what k is $u^k d^{5-k} \geq 1$?

$$k = 3, 4, 5 \implies \text{Prob}(S_5 \geq S_0) = \sum_{k=3}^{5} \binom{5}{k} 0.6^k \, 0.4^{5-k}$$

Some examples

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?
- What is the probability that $S_5 \geqslant S_0$?
- Suppose $S_5 = S_0(1.1)^5$. What is the distribution of S_{10} conditioned on this outcome?

$$S_{10} = S_0 \cdot (1.1)^5 \cdot \begin{cases} 1.1^5 & \text{w/ prob } \binom{5}{5} 0.6^5 \\ 1.1^4 \cdot 0.9 & \text{"} \quad \binom{5}{4} 0.6^4 \cdot 0.4 \\ 1.1^3 0.9^2 & \text{"} \quad \binom{5}{3} 0.6^3 0.4^2 \\ \vdots & \\ 1.1^0 0.9^5 & \text{"} \quad \binom{5}{0} 0.4^5 \end{cases}$$

Some final observations

The process we have defined and investigated has some useful properties:

There are 2^n possible pricing trajectories that the model can take. (2 options at each time.)

But there are only $j+1$ possible outcomes at time t_j: the security price S_j is given by,

$$S_j = S_0 \prod_{q=1}^{j} G_j = S_0 u^{Y_j} d^{j-Y_j},$$

where Y_j is the cumulative number of "heads" realizations in the first j steps:

$$Y_j = \sum_{q=1}^{j} X_q.$$

Since Y_j has exactly $j+1$ possible outcomes, then S_j, whose randomness depends explicitly and only on Y_j, has exactly $j+1$ possible outcomes.

Some final observations

The process we have defined and investigated has some useful properties:

There are 2^n possible pricing trajectories that the model can take. (2 options at each time.)

But there are only $j+1$ possible outcomes at time t_j: the security price S_j is given by,

$$S_j = S_0 \prod_{q=1}^{j} G_j = S_0 u^{Y_j} d^{j-Y_j},$$

where Y_j is the cumulative number of "heads" realizations in the first j steps:

$$Y_j = \sum_{q=1}^{j} X_q.$$

Since Y_j has exactly $j+1$ possible outcomes, then S_j, whose randomness depends explicitly and only on Y_j, has exactly $j+1$ possible outcomes.

Related to the above, the tree is *recombining*, meaning that there are trajectories that reach the same state using different pathways. In particular,

- outcome $(G_1, G_2) = (u, d)$
- outcome $(G_1, G_2) = (d, u)$

yield the same security price $S_2 = udS_0$.

Some final observations

The process we have defined and investigated has some useful properties:

There are 2^n possible pricing trajectories that the model can take. (2 options at each time.)

But there are only $j+1$ possible outcomes at time t_j: the security price S_j is given by,

$$S_j = S_0 \prod_{q=1}^{j} G_j = S_0 u^{Y_j} d^{j-Y_j},$$

where Y_j is the cumulative number of "heads" realizations in the first j steps:

$$Y_j = \sum_{q=1}^{j} X_q.$$

Since Y_j has exactly $j+1$ possible outcomes, then S_j, whose randomness depends explicitly and only on Y_j, has exactly $j+1$ possible outcomes.

Related to the above, the tree is *recombining*, meaning that there are trajectories that reach the same state using different pathways. In particular,

- outcome $(G_1, G_2) = (u, d)$
- outcome $(G_1, G_2) = (d, u)$

yield the same security price $S_2 = udS_0$.

As we saw in the previous example, this is a *Markovian* process: the possibilities at time t_{j+1} can be deduced entirely from the state of things at time t_j.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.