

# Math 5760/6890: Introduction to Mathematical Finance Security Pricing Models

See Petters and Dong 2016, Section 5.1

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*Project: due Tues*

*Oct. 24*

*Thursday: Office hours*

*from 9:30-10:30*

*in WEB 4666*

*(no class)*



# Security pricing

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The remainder of this course focuses on models for pricing securities (and what we can subsequently do with these models).

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The overall goal:

Given a security's time-0 price  $S(0)$ , construct a probabilistic model for  $S(t)$ ,  $t > 0$ .

To accomplish this, we'll need to embed some information about how we expect  $S(t)$  to behave in the future, such as its mean and variance.

## The basic Bernoulli assumption

The underlying idea for this pricing model is as follows:

Consider a 1-period setup, so that we are building a model for  $S(T)$ ,  $T > 0$ .

Our model for  $S(T)$  will be that with some probability  $p$ ,  ~~$S(T)$~~  increases in value, and with probability  $1 - p$ , it decreases in value.

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Hence  $S(T)$  is a random variable, with probability mass function given by,

$$p_{S(T)}(s) = \begin{cases} p, & s = uS(0) \\ 1 - p, & s = dS(0) \end{cases} \quad \begin{array}{l} (uS(0) > S(0)) \\ (dS(0) < S(0)) \end{array}$$

i.e.,

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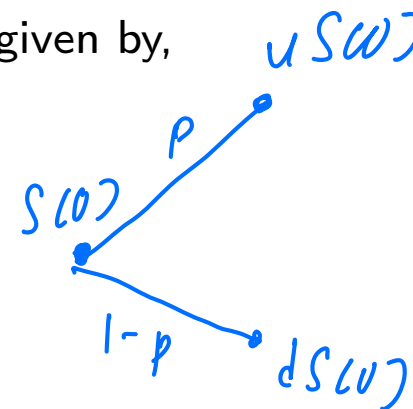
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An alternative (equivalent) point of view: we model the ratio  $S(T)/S(0)$  as the random variable,

$$G := \frac{S(T)}{S(0)} = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1 - p \end{cases}$$

The quantity  $G$  is called the *gross return rate*.

The variable  $S(T)$

$X \sim \text{Bernoulli}(p)$  if

$X = \begin{cases} +1 & \text{w/prob } p \\ 0 & \text{w/prob } 1-p \end{cases} \quad (0 \leq p \leq 1)$

$$S(T) = \begin{cases} uS(0), & \text{with probability } p \\ dS(0), & \text{with probability } 1 - p \end{cases}$$

Thus,  $S(T)$  is a (scaled version of a) **Bernoulli** random variable.

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Clearly for this model to be useful we should have:

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- $d < 1$
- $0 < p < 1$

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This model in isolation has limited utility: it's a 1-period model.

(And is substantially less sophisticated than our portfolio analysis models.)

However, as a building block in a larger scheme, this is a very powerful model.

## Example

Consider a one-period model for the price  $S(T)$  of a security. The time-0 price  $S(0)$  is known, and the predicted mean and variance at time  $T$  is  $(\mu, \sigma^2)$ . Compute the parameters  $(u, d)$  required so that a symmetric Bernoulli model for  $S(T)$  (i.e.,  $p = 1/2$ ) yields the target mean and variance.

$$\mathbb{E} S(T) = \mu$$

$$\text{Var } S(T) = \sigma^2$$

$$S(T) = \begin{cases} uS(0) & \text{w/prob } p \\ dS(0) & \text{w/prob } 1-p. \end{cases}$$

$$\begin{aligned} \mathbb{E} S(T) &= p u S(0) + (1-p) d S(0) \\ &= \frac{1}{2} S(0) (u+d) = \mu \end{aligned}$$

$$\begin{aligned} \text{Var } S(T) &= \mathbb{E} (S(T) - \mu)^2 = p (uS(0) - \mu)^2 \\ &\quad + (1-p) (dS(0) - \mu)^2 \end{aligned}$$

$$= \frac{1}{2} S(\omega)^2 \left[ u - \frac{1}{2}(u+d) \right]^2 + \frac{1}{2} S(\omega)^2 \left[ d - \frac{1}{2}(u+d) \right]^2$$

$$= \frac{1}{2} S(\omega)^2 \left[ \left( \frac{1}{2}u - \frac{1}{2}d \right)^2 + \left( \frac{1}{2}d - \frac{1}{2}u \right)^2 \right]$$

$$= \frac{1}{2} S(\omega)^2 \cdot 2 \left( \frac{1}{2}u - \frac{1}{2}d \right)^2$$

$$= \frac{1}{4} S(\omega)^2 (u-d)^2 = \sigma^2$$

$$\Rightarrow \mu/s(\omega) = \frac{1}{2}(u+d)$$

$$\sigma/s(\omega) = \frac{1}{2}(u-d) \quad (u > d)$$

$$\Rightarrow u = \mu/s(\omega) + \sigma/s(\omega)$$

$$d = \mu/s(\omega) - \sigma/s(\omega)$$

## A 2-period model

This one-step Bernoulli model serves as a building block for a 2-period model:

Let  $(p, u, d)$  be given. Introduce the random variable  $G$  having a shifted Bernoulli distribution:

$$p_G(g) = \begin{cases} p, & g = u \\ 1 - p, & g = d \end{cases}$$

Let  $G_1$  and  $G_2$  be two independent and identically distributed (“iid”) copies of  $G$ .

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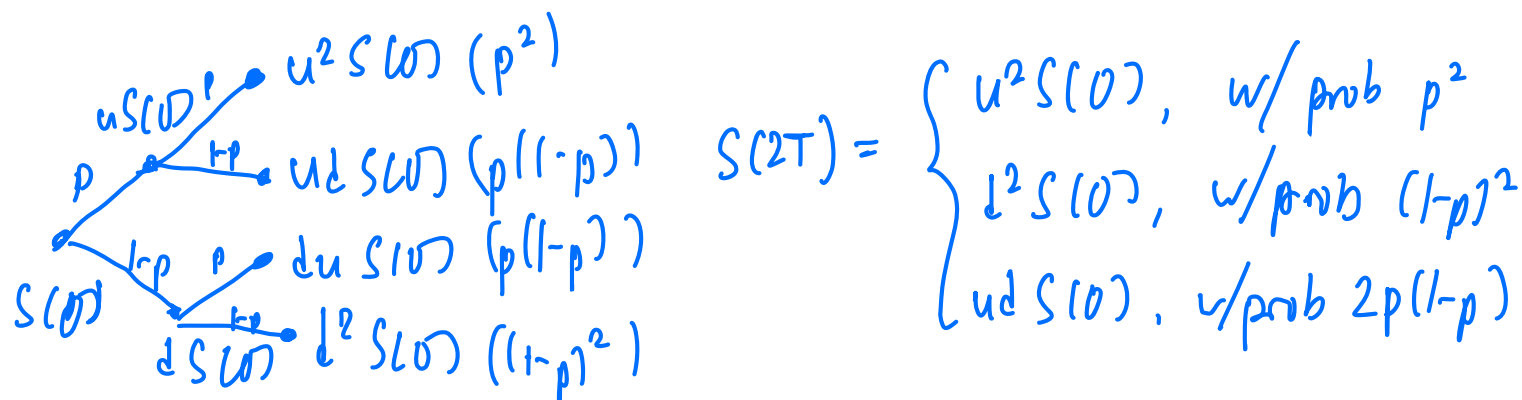
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Our two-period model will define the trajectory of  $S(t)$  as:

$$S(T) = S(0)G_1, \quad S(2T) = S(T)G_2.$$

I.e.,  $S(2T) = G_1 G_2 S(0)$ . This can be visualized as a 4-leaf tree.



## Some notation and terminology

Our 2-period model easily generalizes to the multi-period case, but before doing that we'll introduce some additional notation.

First, we'll now assume that  $T > 0$  is a fixed terminal time (say 1 year), and our multi-period model divides this terminal time up into  $n \in \mathbb{N}$  periods:

- A single period is of time length  $h := T/n$ .
- $t_j := jh = \frac{jT}{n}$  is the time after conclusion of the  $j$ th period. *cash flip*
- $t_0 = 0$ , and  $t_n = T$ .
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For a security  $S(t)$ , its *gross return* (rate) at time  $T$  is given by

$$G := \frac{S(T)}{S(0)}.$$

(Recall that the “standard” or capital-gain return rate is  $R = G - 1$ .)

The *log-return* of  $S(t)$  at time  $T$  is,

$$L := \log G = \log S(T) - \log S(0)$$

$\log = \log_e = \ln$

Our multi-period model can now be introduced: Let  $S$  be a security whose value at time 0,  $S(0)$ , is known. We seek to model the price  $S(T)$  for  $T > 0$  using an  $n$ -period model.



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We can write the time- $n$  price as a product of inter-period gross returns:

$$S(T) = S_n = S_0 \frac{S_1}{S_0} \cdots \frac{S_n}{S_{n-1}} = S_0 \prod_{j=1}^n G_j, \quad G_j := \frac{S_j}{S_{j-1}}.$$

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For our tree based model: we assume that  $\{G_j\}_{j=1}^n$  are iid copies of a (scaled) Bernoulli random variable  $G$ , having distribution:

$$p_G(g) = \begin{cases} p, & g = u \\ 1 - p, & g = d \end{cases} \quad (\text{Assume } u > 1 > d)$$

where we assume knowledge of  $(p, u, d)$  satisfying our previously-mentioned assumptions.

Hence in this model: gross return rates between periods are independent but behave identically (have identical distributions).

# Log-returns in the multi-period model

Since we have

$$S_n = S_0 \prod_{j=1}^n G_j,$$

then this immediately implies that the log-return over the entire time interval  $T$  is the sum of the inter-period log-returns:

$$L = \log \frac{S_n}{S_0} = \sum_{j=1}^n \log G_j =: \sum_{j=1}^n L_j.$$

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This multi-period model is multiplicative in the security price, but additive in the log-return.

And note that the  $L_j$  are also iid, with distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p, \\ \log d, & \text{with probability } 1 - p. \end{cases}$$

$\log u > 0$   
 $\log d < 0$


Hence, the  $L_j$  are also (scaled versions of) Bernoulli random variables, and the cumulative log-return is a sum of iid Bernoulli random variables.

This model:

- Specifies the random distribution of  $S(T)$  through  $n$  intermediate steps/periods.
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- Specifies the random distribution of  $S(T)$  through  $n$  intermediate steps/periods.
- The relative (ratio) change between successive periods has a fixed distribution.
- All changes between periods are independent of each other.
- The tuple  $(p, u, d)$  describes the distribution of the gross return between periods.
- This results in a relatively complicated model: there are  $2^n$  possible outcomes for  $n$  periods.
- Some (many) of the  $2^n$  paths result in the same end price of  $S(T)$ .
- At some point we'll want to relate the (standard) return rate  $r$  of  $S(T)$  to  $(p, u, d)$ .
- Next time:  $S(T)$  has a **Binomial** random variable distribution. Hence, this model is called the *Binomial Pricing Model*.

 Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.