Math 5760/6890: Introduction to Mathematical Finance
Risk Measures

See Petters and Dong 2016, Section 4.2

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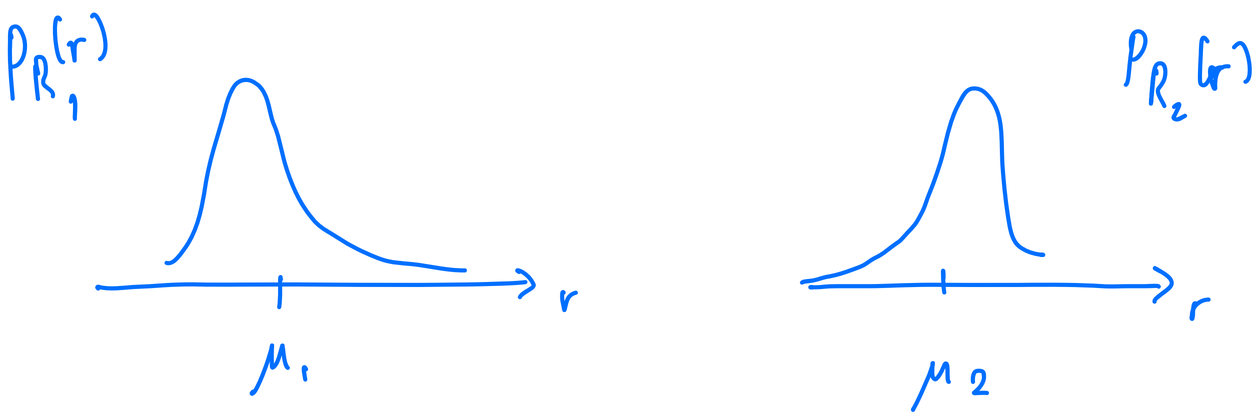
October 5, 2023

Portfolio risk

Until now, our definition for *risk* has been the standard deviation/variance ("spread") of a return rate.

This is, necessarily, a deceptive measure of the colloquial notion of "risk".
For example: sometimes there is a clear preference between two options with identical mean and variance.



$\mu_1 = \mu_2$

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Note: if R is a Gaussian RV, then μ, σ are sufficient to understand risk.

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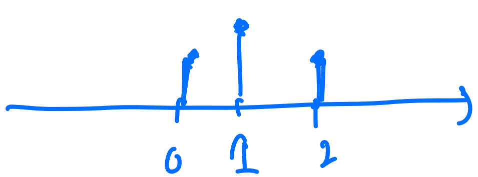
This depends quite a bit on how one *qualitatively* defines risk.

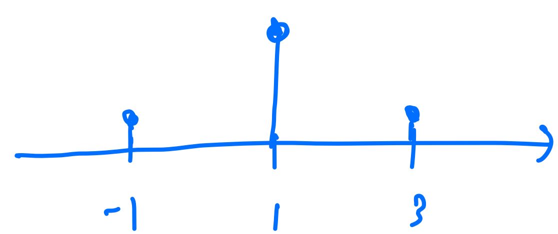
Example

Let R_1 and R_2 be discrete random variables with mass functions given by,

$$p_{R_1}(r) = \begin{cases} \frac{1}{2}, & r = 1, \\ \frac{1}{4}, & r = 0, \\ \frac{1}{4}, & r = 2. \end{cases} \qquad p_{R_2}(r) = \begin{cases} \frac{3}{4}, & r = 1, \\ \frac{1}{8}, & r = -1, \\ \frac{1}{8}, & r = 3. \end{cases}$$

Would you prefer a portfolio with return R_1 or R_2?





Measures of risk

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- Sharpe ratio
- Sortino ratio
- Treynor ratio
- The (Jensen's) alpha
- C/VaR
- ⋮

None of these metrics is always "better" than another, but many have dis/advantages compared to others.

Ratios

To discuss some risk metrics, let's review some notation:

- R: the (random) return rate of a security (which could be a portfolio)
- r: the deterministic capital market risk-free rate
- R_M: the market portfolio's return rate
- (μ, σ^2): $(\mathbb{E}R, \mathrm{Var}\, R)$
- (μ_M, σ_M^2): $(\mathbb{E}R_M, \mathrm{Var}\, R_M)$
- β: $\rho(R, R_M)\frac{\sigma}{\sigma_M}$

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The first ratio we'll consider is the **Sharpe ratio**, defined as,

$$\mathrm{Sh}(R) := \frac{\mu - r}{\sigma}$$

If the risk-free rate r is not determnistic, then the denominator should be the standard deviation of $R - r$.

In most simplified cases, this ratio is the slope of the security's capital allocation line.

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A second ratio is the **Sortino ratio**:

$$\mathrm{So}(R) := \frac{\mu - t}{\sigma_-(t)}$$

Above, t is a target return (e.g., the risk-free rate r). The quantity $\sigma_-^2(t)$ is the semivariance, or the "downside deviation" from the target t.

$$\sigma_-^2(t) = \int_{-\infty}^{t} (r-t)^2 p_R(r)\, dr$$ ← spread, if the spread is negative.

$$\sigma_-^2(\infty) = \mathrm{Var}\, R.$$

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Yet another option: the **Treynor ratio**:

$$\mathrm{Tr}(R) := \frac{\mu - r}{\beta}$$

Recall that the β metric measures volatility relative to how R tracks with the market: such market-related risk is called *systematic risk*. Hence, the Treynor ratio is a reward-risk ratio, where "market-related risk" is used.

The "alpha"

Yet another measure of risk is the "alpha" of a security, which measures premium relative to the capital asset pricing model.

For example, "Jensen's alpha" is defined as,

$$\alpha = (\mu - r) - \beta \left(\mu_M - r\right),$$

where the right-hand side is zero in theory, but not in practice.

Again, this is a return relative to the market.

Value at Risk

A random variable L has a cumulative distribution function:

$$F_L(\ell) = P(L \leqslant \ell), \qquad\qquad F_L : \mathbb{R} \to [0,1]$$

The *quantile function* for L is the functional inverse of F_L:

$$Q_L(p) \coloneqq F_L^{-1}(p) = \min \left\{ \ell \in \mathbb{R} \;\middle|\; F_L(\ell) \geqslant p \right\}, \qquad\qquad Q_L : [0,1] \to \mathbb{R}.$$

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$$\mathrm{VaR}_p(L) := Q_L(p).$$

For example, $\mathrm{VaR}_p(R) = -0.4$ when $p = 0.01$, this means that with 1% probability, R will be at most -40%.

If one assumes normality of random variables, value at risk is straightforward to compute using the *probit function*.

Conditional value at risk

Value at risk is a relatively nuanced concept: if you can compute VaR for arbitrary p, you know *everything* about a random variable.

Hence, this is powerful, but can be difficult to transparently analyze since VaR for a single p value can be informative, but is a limited picture.

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An even more nuanced quantity involved value at risk is the *conditional value at risk*, which is the expectation conditioned on a VaR event:

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Warning: Sometimes VaR and CVaR are written in terms of the loss. I.e., $\mathrm{VaR}_p(R) = -0.4$ will be written as the p-VaR of R at $p = 1\%$ is 40%.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.