Math 5760/6890: Introduction to Mathematical Finance
Capital Asset Pricing Model

See Petters and Dong 2016, Section 4.1

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Capital Markets with Markowitz Portfolios

We are interested in building portfolios from weighted combinations of risky and a riskless asset:

- We have access to N risky securities, e.g., stocks. We've used standard Markowitz portfolio analysis to build efficient and optimal portfolios.
- With only risky securities, the efficient frontier is the upper half of the graph of a hyperbola.

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- In capital markets, a(n essentially) riskless asset with some fixed risk-free return rate r is available.
- Capital market theory demonstrates that the efficient frontier is a one-sided line connecting the risky security with the *market portfolio*, i.e., a unique efficient Markowitz portfolio.

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We'll now discuss the capital asset pricing model, which is a useful tool in pricing securities.

Risk premiums

Risk premiums are simple ways to characterize the payoff for taking on risk.

Consider a captial market setup: there is a risk-free rate r, and risky securities, and a market portfolio having stochastic return R_M, with risk and expected return (σ_M, μ_M), respectively.
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Let R be the return rate of a (risky) security. (Perhaps it's one of the securities on the market.)

This security has its own risk-return profile (σ, μ).

The *risk premium* k of this security is

$$k = \mu - r,$$

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Every security has a premium, even the market portfolio:

$$k_M := \mu_M - r. \quad > 0$$

The *beta* metric

In the same capital market, we require the introduction of a particular risk metric for a security, its *beta*.

The "beta" of a security is a statistical measure of how much a security correlates with the market portfolio,

$$\beta := \frac{\mathrm{Cov}(R, R_M)}{\sigma_M^2} = \frac{\mathrm{Cov}(R, R_M)}{\sigma \sigma_M} \frac{\sigma}{\sigma_M} = \rho(R, R_M) \frac{\sigma}{\sigma_M}.$$

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For example, this definition shows that β is a lower bound for the risk of the security R *relative* to the market portfolio risk:

$$|\beta| = |\rho(R, R_M)| \frac{\sigma}{\sigma_M} \leqslant \frac{\sigma}{\sigma_M}$$

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In principle, the market portfolio also has a beta:

$$\beta_M = \frac{\mathrm{Cov}(R_M, R_M)}{\sigma_M^2} = 1.$$

The capital asset pricing theorem

We have two "measures" of a given security return rate R relative to the market:

- The security's risk premium k, which measures its advantage (relative to the market) for taking on risk.
- The security's beta β, which measures its volatility (relative to the market).

One suspects that these notions should be quantitatively relatable.

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Theorem (Capital asset pricing theorem)

The risk premium k and beta β of a risky security are related to the market risk premium k_M via

$$k = \beta k_M.$$

I.e., this is,

$$\mu - r = \beta(\mu_M - r).$$

Proof idea: $\beta = \frac{\mathrm{Cov}(R, R_M)}{\sigma_M^2} = \frac{\underline{w}^T \underline{\underline{A}}\, \underline{w}_M}{\underline{w}_M^T \underline{\underline{A}}\, \underline{w}_M} = \cdots\cdots = \frac{\mu - r}{\mu_M - r}$

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The *Capital Asset Pricing Model* assesses prices on securities (e.g., through analysis of their risk premiums) through the theorem above.

The capital asset pricing model

$$\mu - r = \beta(\mu_M - r).$$

Some immediate consequences:

- A security's premium is *directy proportional* to its beta. The proportionality constant is just the market risk premium.
- $\beta < 0$: the security's premium behaves in an opposite manner to the market. (E.g., this is a good candidate for hedging.)
- $0 \leqslant \beta \leqslant 1$: the security tracks in the same direction as the market, with smaller or equal volatility.
- $\beta > 1$: the security tracks with the market, but has higher volatility (and higher premium).

Security pricing

$$\mu - r = \beta(\mu_M - r)$$

Example

Consider a per-unit stock price S. Given the risk-free rate r and statistics for $S(T)$ (its expectation and beta), what should today's price $S(0)$ be?

$$R(T) = \frac{S(T) - S(0)}{S(0)} \implies S(0) = \frac{S(T)}{1 + R(T)}$$

$$\Downarrow \mathbb{E}$$

$$\mathbb{E}R(T) = \frac{\mathbb{E}S(T) - S(0)}{S(0)} \longrightarrow S(0) = \frac{\mathbb{E}S(T)}{1 + \underbrace{\mathbb{E}R(T)}_{\mu}} = \frac{\mathbb{E}S(T)}{1 + r + \beta(\mu_M - r)}$$

The security market line

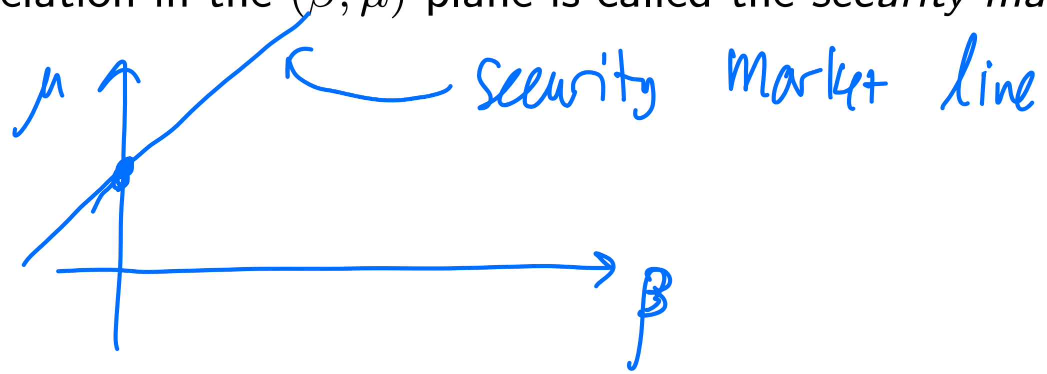
Because the beta is essentially a (properly scaled) proxy for σ, it is common to consider plots in the (β, μ) plane.

The capital asset pricing model is given by,

$$\mu = r + \beta(\mu_M - r),$$

which is a linear relation. (assume $\mu_M > r$)

The graph of this relation in the (β, μ) plane is called the *security market line*.



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The security market line is one of the more useful tools for qualitatively determining valuation of stocks: Given a stock's β, if

- the expected return μ lies above the security market line, then the stock is overvalued.
- the expected return μ lies below the security market line, then the stock is undervalued.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.