Math 5760/6890: Introduction to Mathematical Finance
Capital Market Theory

See Petters and Dong 2016, Section 4.1

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Markowitz Portfolio Analysis

Given N securities, Markowitz Portfolio Analysis is a one-period model that prescribes efficient portfolios

- as those for which one cannot attain higher reward without higher risk
- as those for which one cannot reduce risk without also reducing reward

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$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \;\text{ subject to }\; \begin{aligned} \langle \boldsymbol{w}, \mathbf{1} \rangle &= 1, \text{ and} \\ \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle &= \mu_P. \end{aligned}$$

where

- $\boldsymbol{A} = \mathrm{Cov}(\overset{R}{\cancel{R_P}})$ is assumed positive-definite
- $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ is assumed <u>not</u> parallel to $\mathbf{1}$.
- μ_P is a target expected return rate

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- $\boldsymbol{A} = \mathrm{Cov}(R_P)$ is assumed positive-definite
- $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ is assumed <u>not</u> parallel to $\boldsymbol{1}$. (practically)
- μ_P is a target expected return rate

All of this involves risky securities. In practice, in particular in *capital markets*, risk-free securities are available.

Capital markets

A capital market is a financial market where securities can be purchased and sold. This includes stocks, bonds, and other underwritten debt instruments.

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In the capital market, investors have access to practically zero-risk securities with risk-free interest rates, such as government bonds.

Our Markowitz theory really only applies to risky securities.

Capital Market (portfolio) Theory augments Markowitz portfolio theory by including the availability of a risk-free security.

The risk-free security

To fit a risk-free security in the context of our Markowitz model:

- Let $R_0(t)$ be the return rate of the risk-free asset. (We'll immediately speak in terms of rates and not per-unit asset price.)
- In the one-period model with period $T > 0$, the return rate is $R_0(T) = r$, where $r > 0$ is a deterministic constant, the *risk-free rate*.
- I.e., r is the return rate in time units corresponding to T.
- For example: a bond with r playing a role similar to yield to maturity

Assuming a single riskless security is sufficient:

If R_{-1} and R_0 are (return rates of) riskless securities, then any linear comb. of R_0 and R_{-1} is a single riskless security.

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- We allocate weight w_0 to the risk-free security. $w_0 > 0$ corresponds to investing in the security.
- $w_0 < 0$ (effectively) corresponds to borrowing money at the risk-free rate r.
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- $w_0 < 0$ is not really shorting the security – instead the investor hopes to get better-than-r return rate using the borrowed capital.
- It is only reasonable to ask for a target portfolio expected return rate μ_P satisfying $\mu_P \geqslant r$.
- Similarly, it is irrational for an investor to borrow the risk-free security to invest in a lower-return risky portfolio.

The augmented Markowitz setup

We set up the same problem as in the N-security Markowitz case:

$$\widetilde{\boldsymbol{R}} = \begin{pmatrix} R_0 \\ \boldsymbol{R} \end{pmatrix} = \begin{pmatrix} R_0 \\ R_1 \\ \vdots \\ R_N \end{pmatrix} \in \mathbb{R}^{N+1}, \qquad \widetilde{\boldsymbol{w}} = \begin{pmatrix} w_0 \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix} \in \mathbb{R}^{N+1},$$

where $\widetilde{\boldsymbol{R}}$ has statistics:

$$\widetilde{\boldsymbol{\mu}} := \begin{pmatrix} r \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} r \\ \mathbb{E}R_1 \\ \mathbb{E}R_2 \\ \vdots \\ \mathbb{E}R_N \end{pmatrix} \in \mathbb{R}^{N+1}, \mathrm{Cov}(\widetilde{\boldsymbol{R}}) = \widetilde{\boldsymbol{A}} = \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix}, \quad \boldsymbol{A} = \mathrm{Cov}(\widetilde{\boldsymbol{R}}).$$

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We'll assume again that $\widetilde{\boldsymbol{\mu}}$ is not parallel to $\boldsymbol{1}$, and that $\boldsymbol{A}$ is positive-definite.

Note, however, that $\mathrm{Cov}(\widetilde{\boldsymbol{R}})$ is <u>not</u> positive-definite.

The optimization problem

We can now formulate the optimization problem:

$$\min_{\widetilde{\boldsymbol{w}}} \widetilde{\boldsymbol{w}}^T \widetilde{\boldsymbol{A}} \widetilde{\boldsymbol{w}} \text{ subject to } \langle \widetilde{\boldsymbol{w}}, \mathbf{1} \rangle = 1, \text{ and}$$
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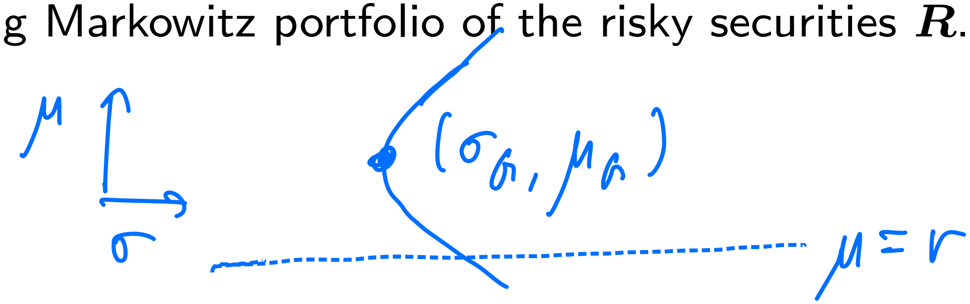
- If $\mu_P < r$, then any portfolio we compute is not efficient.
- It is also reasonable to assume that $\mu_G > r$, where μ_G is the expected return of the global variance-minimizing Markowitz portfolio of the risky securities $\boldsymbol{R}$.



Some intuition

However this optimization problem turns out, we know that the return rate of the resulting portfolio will have the form,

$$\langle \underline{\tilde{w}}, \underline{\tilde{R}} \rangle = \tilde{R}_p = w_0 R_0 + \langle \boldsymbol{w}, \boldsymbol{R} \rangle,$$

i.e., this will be a linear combination of a riskless asset (R_0) along with a risky asset (R_1).

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More generally, note that since $\langle \tilde{\boldsymbol{w}}, \mathbf{1} \rangle = 1$, then the above can be written as,

$$w_0 R_0 + (1 - w_0) \left\langle \frac{\boldsymbol{w}}{1 - w_0}, \boldsymbol{R} \right\rangle ,$$

$\langle \tilde{w}, \mathbb{1} \rangle = 1$

where

$$\left\langle \frac{\boldsymbol{w}}{1 - w_0}, \mathbf{1} \right\rangle = \frac{1}{1 - w_0} \sum_{i=1}^{N} w_i = 1.$$

$w_0 + w_1 + \cdots + w_N = 1$

$\frac{w_1 + w_2 + \cdots w_N}{1 - w_0} = 1$

(assuming $w_0 \neq 1$).

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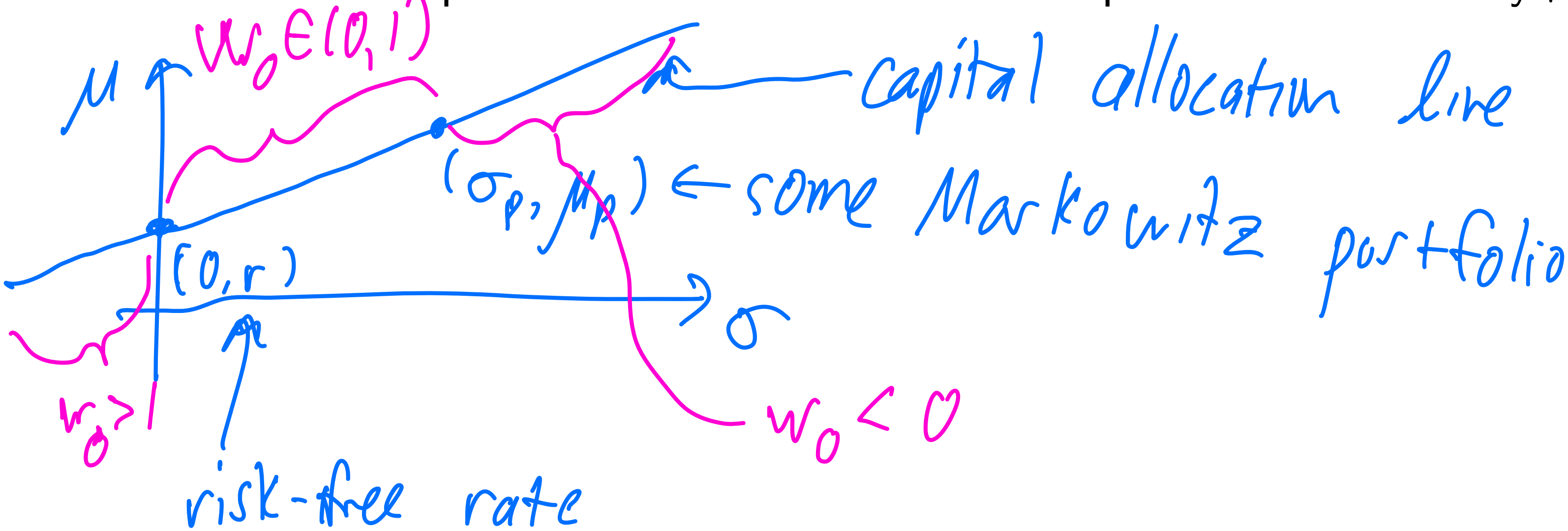
Hence, our portfolio will always be a linear combination of a riskless asset and a risky Markowitz portfolio.

Capital allocation lines

The set of points in the (σ, μ) plane corresponding to linear combinations of a fixed riskless asset and a fixed risky Markowitz portfolio is a **~~captial~~ capital allocation line**.

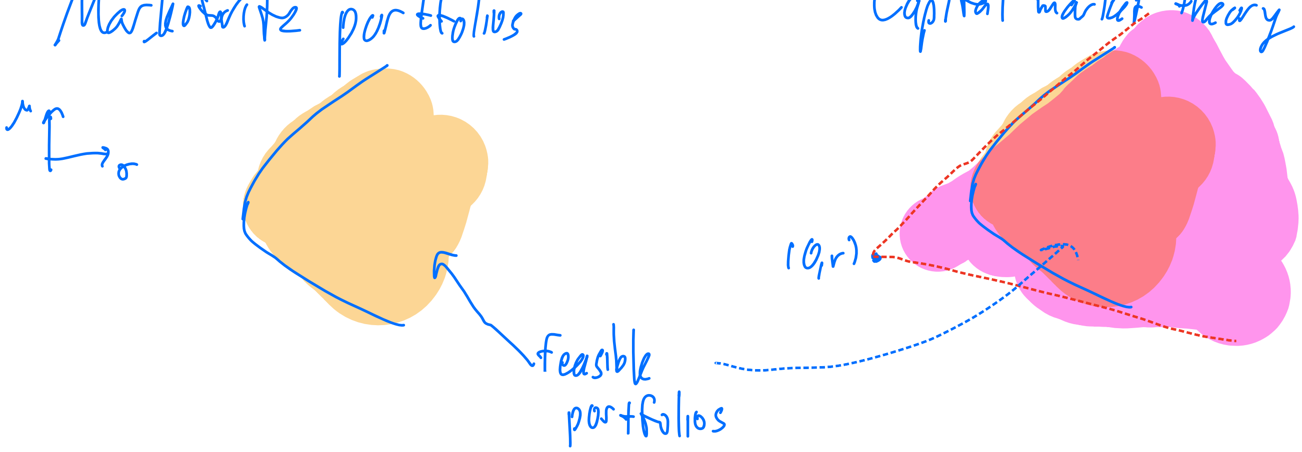
These correspond to risk-return tradeoffs when combining a risky and riskless asset.

Points on the capital allocation lines are feasible portfolios in the risky+riskless setup.



Markowitz portfolios

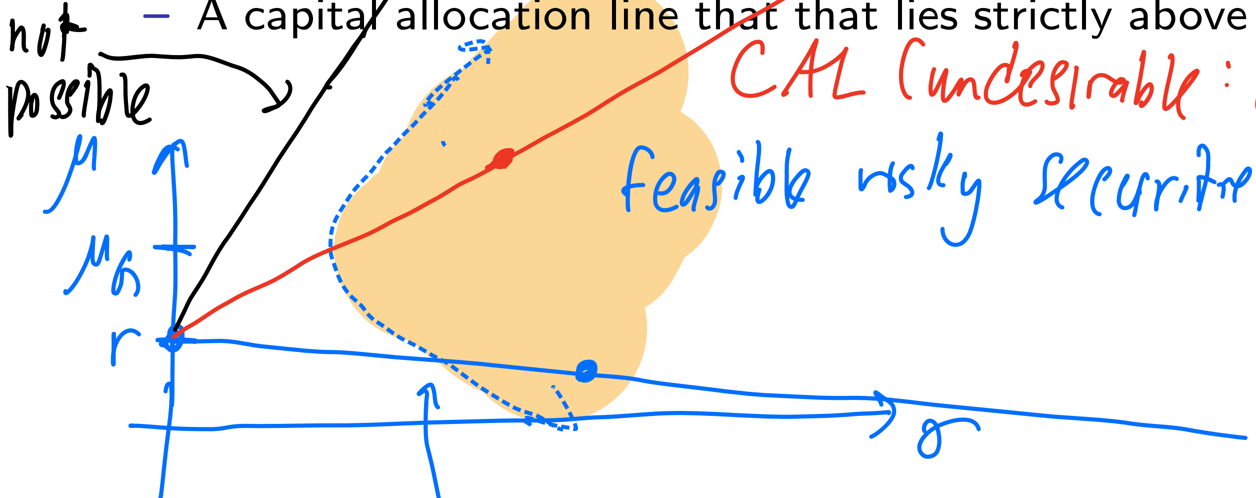
Capital market theory



Solution to the optimization problem

With all the above understanding, we know that any solution to our augmented portfolio problem will lie on a capital allocation line. Which capital allocation line will be involved?

- A capital allocation line sloping downward can't possibly be of interest.
- A capital allocation line that cuts through the Markowitz bullet can't correspond to efficient portfolios
- A capital allocation line that that lies strictly above the Markowitz bullet isn't possible.



CAL (undesirable: any point on this line has other feasible points elsewhere w/ higher μ + lower σ)

CAL (undesirable, because increasing σ decreases μ).

risk-optimal frontier for the risky Markowitz setup

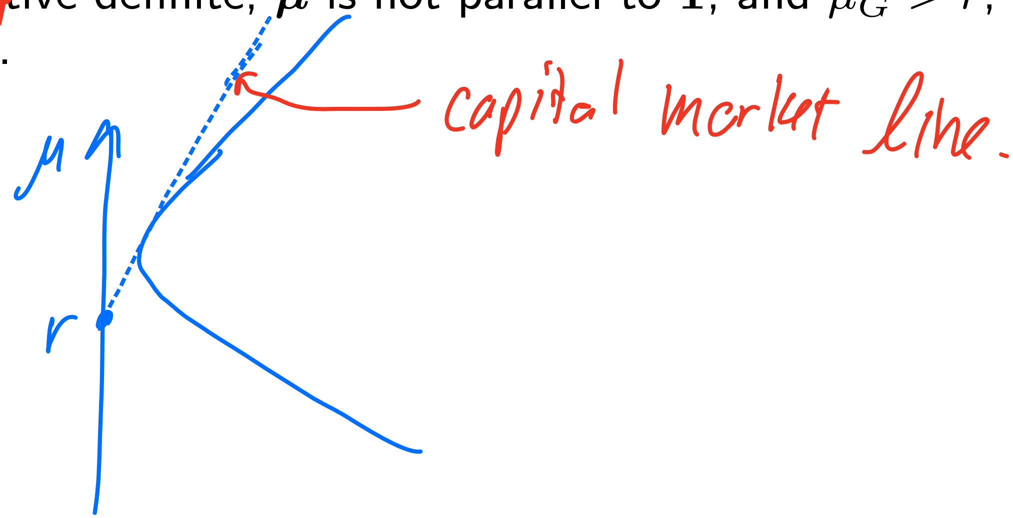
Only option left: a CAL tangent to upper half of the Markowitz bullet.

→ This will be our solution to the opt. problem.

The ~~captial~~ Capital market line

There is a unique capital allocation line that provides maximum expected return vs risk: this is the **~~captial~~ capital market line**.

- If $\boldsymbol{A}$ is positive-definite, $\boldsymbol{\mu}$ is not parallel to $\mathbf{1}$, and $\mu_G > r$, there is a unique capital market line.



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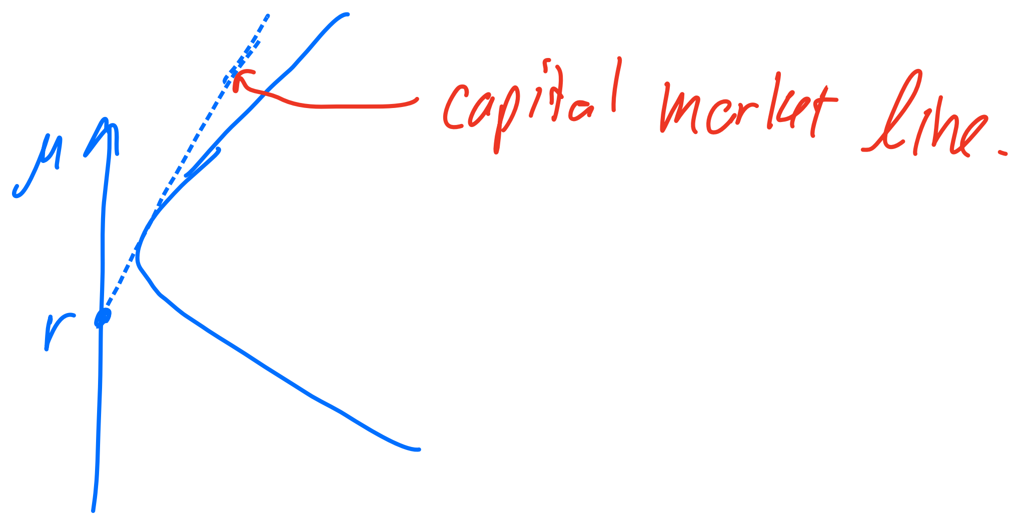
- The capital market line is the unique upward-sloping tangent line to the risky Markowitz efficient frontier that passes through the riskless security at $(\sigma, \mu) = (0, r)$.



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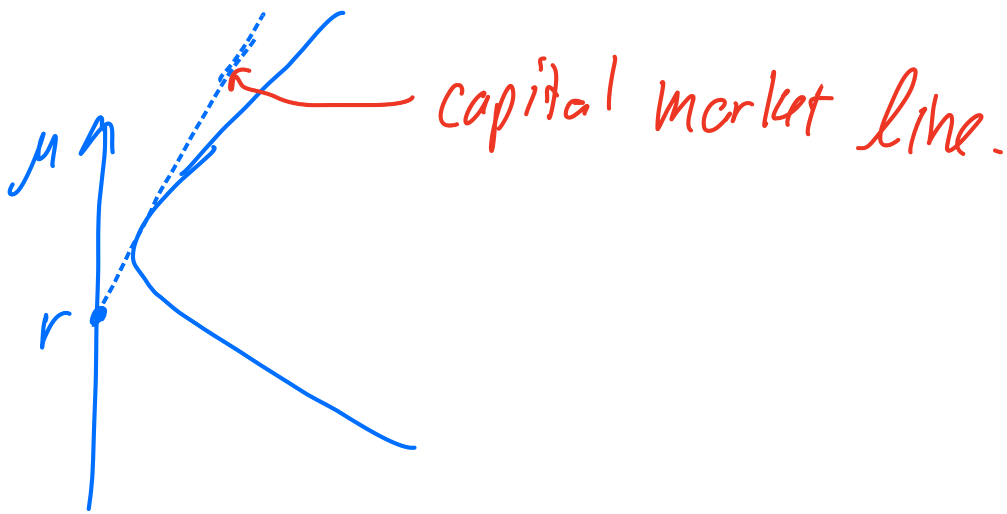

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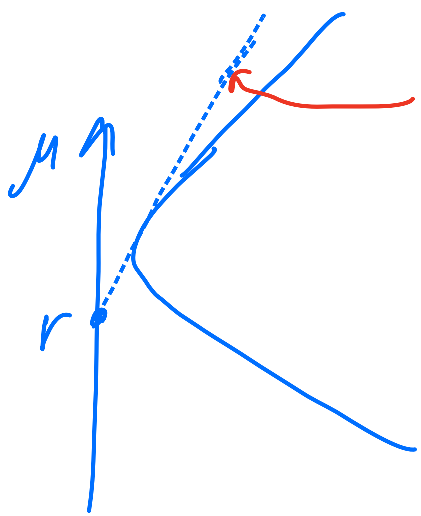
- The capital market line for $\sigma \geqslant 0$ is the efficient frontier for this optimization problem.



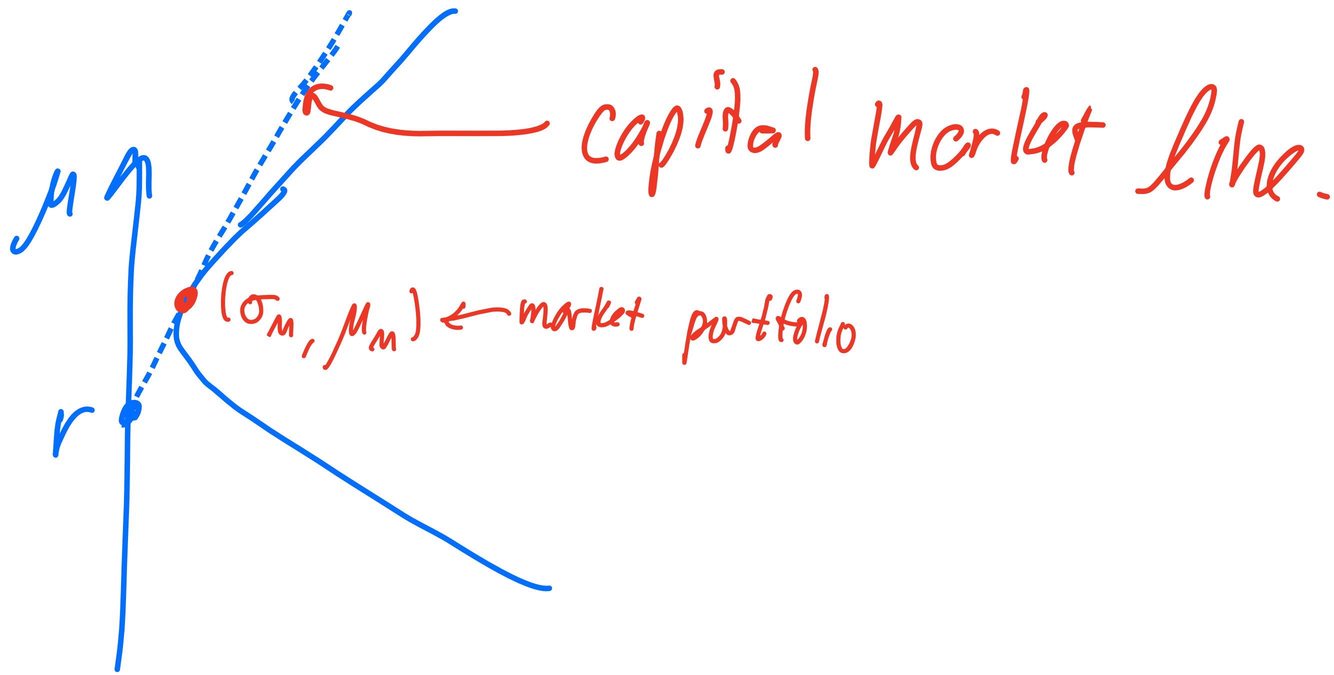
capital market line.

(solution to the augmented Markowitz problem).

The market portfolio

The point of tangency of the capital market line to the risky Markowitz efficient frontier corresponds to the **market portfolio**; it is part of the risky efficient frontier.

The market portfolio is the most desired risky portfolio for investors: It is the optimal risky asset for an investor to hold, assuming ability to invest in the riskless asset.



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Let $\boldsymbol{w}_M \in \mathbb{R}^N$ denote the market portfolio, with risk+return (σ_M, μ_M).

Recall that we invest $w_0 \leqslant 1$ into the riskless asset. Therefore, the full portfolio we invest in corresponds to:

$$\mathbb{R}^{N+1} \ni \widetilde{\boldsymbol{w}} = \underbrace{w_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{Riskless}} + \underbrace{(1-w_0)\begin{pmatrix} 0 \\ \boldsymbol{w}_M \end{pmatrix}}_{\text{Risky/Markowitz}}, \qquad \widetilde{R}_P = \left\langle \widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{R}} \right\rangle$$

$\in \mathbb{R}^N$

This immediately reveals statistics of this portfolio:

$$\mathbb{E}\widetilde{R}_P = \mathbb{E}\left\langle \boldsymbol{w}, \widetilde{\boldsymbol{R}} \right\rangle = w_0 r + (1-w_0)\mu_M$$

$$\text{Var}\widetilde{R}_P = (1-w_0)^2 \text{Var}\left\langle \boldsymbol{w}, \boldsymbol{R} \right\rangle = (1-w_0)^2 \sigma_M^2.$$

Borrowing the riskless security

There are two regimes of interest on the capital market line:

- The portion between $(0, r)$ and the market portfolio corresponds to investing in the riskless asset ($w_0 > 0$).
- The portion above the market portfolio corresponds to borrowing against the riskless asset ($w_0 < 0$).

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.