Math 5760/6890: Introduction to Mathematical Finance
Mutual Fund Theorem

See Petters and Dong 2016, Section 3.5

Thursday's class (Sept. 28 online on zoom)

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HW #5: Due Thursday (Oct 5)

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Markowitz Portfolio Analysis

Given N securities, Markowitz Portfolio Analysis is a one-period model that prescribes efficient portfolios

- as those for which one cannot attain higher reward without higher risk
- as those for which one cannot reduce risk without also reducing reward

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The definition of reward and risk are first- and second-order statistics (mean-variance analysis) of the portfolio return rate. The risk-optimal (not necessarily efficient) formulation is

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \text{ subject to } \langle \boldsymbol{w}, \mathbf{1} \rangle = 1, \text{ and}$$

$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

where

- $\boldsymbol{A} = \text{Cov}(R_P)$ is assumed positive-definite
- $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ is assumed not parallel to $\mathbf{1}$.
- μ_P is a target expected return rate

Practical concerns

Our setup assumes statistics such as $\mathbb{E}\boldsymbol{R}$ and $\mathrm{Cov}(\boldsymbol{R})$ are available.

These statistics are typically computed using historical data. E.g.,

- `https://www.portfoliovisualizer.com`
- `https://www.buyupside.com/index.html`
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Typically, cross-covariance entries are reported as *correlations*. The main reason for this is that correlations aren't sensitive to units of time (scaling):

$$\mathrm{Cov}(aX, bY) = ab\mathrm{Cov}(X, Y)$$

$$\sqrt{\mathrm{Var}(aX)} = |a|\sqrt{\mathrm{Var} X}$$

$$\Downarrow$$

$$\rho(aX, bY) := \frac{\mathrm{Cov}(aX, bY)}{\sqrt{\mathrm{Var}(aX)\ \mathrm{Var(bY)}}} = \frac{ab}{|ab|} \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var(Y)}}} = \mathrm{sign}(ab)\rho(X, Y).$$

The model assumptions

A couple of small digressions:

- The assumption about positive-definite covariance is quite natural: in practice one never sees completely correlated securities.
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  However, if it were to happen, then one can show:
  - The expected return rate for any Markowitz portfolio can only be the value μ_1.
  - Hence, the set of all feasible portfolios collapses to a line on a μ_P vs σ_P plot.
  - The (unique) global variance-minimizing portfolio is the only rational choice.

Mutual Funds

A simple example of portfolio engineering in practice: mutual funds.

A mutual fund is an investment fund in which assets from various investors are collectively pooled to purchase securities according to a portfolio design.

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Generally speaking, there are actively-managed, and passively-managed funds.

- Passively managed funds purchase stocks according to some preset rule, such as according to a stock market index ("index funds").
  This requires relatively little hands-on work, and so typically have lower fees.

- Actively managed funds have financial professionals who attempt to engineer portfolios to beat preset rules/market index returns.
  Since this requires employees/managers to engineer portfolios, these typically come with higher fees.

(Most actively managed funds are built with procedures that are substantially more technical than simple mean-variance analysis.)

Building N-security portfolios

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Hence, in practice one requires nontrivial investment + infrastructure to construct N-security portfolios.
This is essentially why actively managed funds charge fees – and so it is reasonable to assume that actively managed funds produce efficient portfolios.

The little guy

This whole reality produces a quandry for you and me:

- It's best to leave construction of N-security portfolios to professionals, since most of us don't have the time + resources to do this ourselves.
- The available professionally-managed funds might not have an expected return rate μ_P that we prefer.

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There is a reasonable solution here, but let's clarify the setup. We'll assume:

- We are in an N-security market, and all (actively-managed) mutual funds invest in the same N securities.
- Actively-managed mutual funds produce efficient portfolios. → portfolios are on efficient frontier.

With this setup, then a(ny) mutual fund will produce a portfolio given by,

$$\boldsymbol{w} = \boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1,$$

for some target μ_P, with $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ some computable vectors.

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Using this, we can construct other efficient portfolios with arbitrary values of $\mathbb{E}R_P$.

Assume I have access to portfolios α, β with target expected return rates μ_α, μ_β.

$$\underline{w}_\alpha = \underline{v}_0 + \mu_\alpha \underline{v}_1$$
$$\underline{w}_\beta = \underline{v}_0 + \mu_\beta \underline{v}_1$$

} What if I want a portfolio w/ return rate $\mu = \mathbb{E} R_p$?

Choose γ, and construct a portfolio as

$$\underline{w} = \gamma \underline{w}_\alpha + (1-\gamma)\underline{w}_\beta = \gamma \underline{v}_0 + \gamma \mu_\alpha \underline{v}_1 + (1-\gamma)\underline{v}_0 + (1-\gamma)\mu_\beta \underline{v}_1$$

$$= \underline{v}_0 + \underbrace{(\gamma \mu_\alpha + (1-\gamma)\mu_\beta)}\underline{v}_1$$

Choose γ to make this equal μ.

$$\Longrightarrow \mu = \gamma \mu_\alpha + (1-\gamma)\mu_\beta$$

$$\gamma = \frac{\mu - \mu_\beta}{\mu_\alpha - \mu_\beta} \quad (\mu_\alpha \neq \mu_\beta)$$

The Mutual Fund Theorem

We have proven the following:

Theorem (Mutual Fund Thm)

Consider an N-security market. If $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$ are two distinct efficient portfolios (e.g., mutual funds), then any other efficient portfolio $\boldsymbol{w}$ can be written as,

$$\boldsymbol{w} = \alpha \boldsymbol{w}_\alpha + (1-\alpha)\boldsymbol{w}_\beta,$$

where $\boldsymbol{w}_\alpha$ and $\boldsymbol{w}_\beta$ are two other distinct efficient portfolios (e.g., mutual funds).

With R_P, R_α, and R_β the return rates corresponding to portfolios $\boldsymbol{w}$, $\boldsymbol{w}_\alpha$, and $\boldsymbol{w}_\beta$, respectively, then if $\mathbb{E}R_P$ is between $\mathbb{E}R_a$ and $\mathbb{E}R_b$, then both coefficients α and $1-\alpha$ are non-negative.

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The punchline: if we have the ability to purchase from two distinct efficient mutual funds (N-security portfolios), then we can easily construct a 2-security portfolio with a target mean that has the risk characteristics of the whole set of N securities.

I.e.:

$$(\text{2 efficient } N\text{-security portfolios}) \Longrightarrow (\text{All efficient } N\text{-security portfolios})$$

("Efficient" can be replaced by "risk-optimal")

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.