Math 5760/6890: Introduction to Mathematical Finance
The N-security Markowitz Portfolio

See Petters and Dong 2016, Section 3.3-3.4

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A quick recap

For Markowitz 2-security portfolio optimization:

- Return rates $\boldsymbol{R}$ for the two securities are random variables
- Assume first- and second-order statistics of these are available: $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ and $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$ (Book: $V = \mathrm{cov}(R)$)
- The portfolio is defined by the weights $\boldsymbol{w}$

Return rate: $R_p = \langle w, R \rangle$

- The expected return rate of the portfolio is $\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle$
- The squared risk of the portfolio is $\sigma_P^2 = \mathrm{Var}\langle \boldsymbol{R}, \boldsymbol{w} \rangle = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w}$

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- The portfolio is defined by the weights $\boldsymbol{w}$
- The expected return rate of the portfolio is $\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle$
- The squared risk of the portfolio is $\sigma_P^2 = \mathrm{Var}\langle \boldsymbol{R}, \boldsymbol{w} \rangle = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w}$
- Generally, no optimization is needed for 2-security portfolios: the weight constraints uniquely identify portfolios
- Many risk-optimal portfolios are *not* efficient
- The set of risk-optimal portfolios can be explicitly parameterized + plotted
- There is a global variance-minimizing portfolio – it's possible investors might not want this portfolio.
- The *efficient frontier* can be "easily" identified.

The N-security Markowitz portfolio

Recall: $\sigma_p^2 = w^T A w$

We now consider the risk-minimization problem for an N-security portfolio:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \ \text{ subject to } \ \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1, \text{ and}$$
$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

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- The constraints are the same: we enforce $\boldsymbol{w}$ is a valid portfolio weight, and that the portfolio has expected return rate μ_P.
- Just like before, we will see that risk-optimal portfolios need not be efficient ones.
- We make assumptions that are, by now, commonplace:
  - $\boldsymbol{A}$ is positive-definite
  - $\boldsymbol{\mu}$ is not parallel to $\mathbf{1}$
  - We allow short selling

X, Y : scalar RV's

$$\underline{Z} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$Cov(X,X) = Var(X)$$

$$Cov(\underline{Z}) = \begin{pmatrix} Cov(Z_1, Z_1) & Cov(Z_1, Z_2) \\ Cov(Z_2, Z_1) & Cov(Z_2, Z_2) \end{pmatrix}$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \qquad \sigma_X^2 = Var\,X$$

$$\underline{\underline{\Sigma}}^{(Z)} = diag(\sigma_{Z_1}, \sigma_{Z_2} \ldots \sigma_{Z_N})$$

$$\underline{\underline{\Sigma}}^{-1}\, Cov(Z)\, \underline{\underline{\Sigma}}^{-1} \Leftarrow \text{matrix of correlation coefficients.}$$

Computing the risk-optimal portfolio

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \;\text{ subject to }\; \langle \boldsymbol{w}, \mathbf{1} \rangle = 1, \text{ and}$$
$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

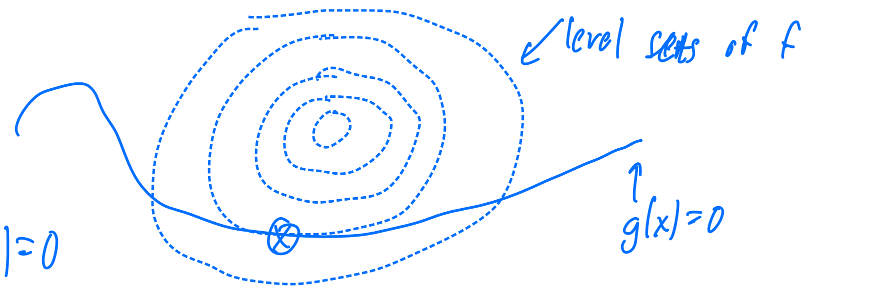
We assume $\boldsymbol{A}$ is positive-definite and $\boldsymbol{\mu}$ is not parallel to $\mathbf{1}$.

Unlike the 2-security case, we have a real optimization problem: the two linear constraints are not sufficient to uniquely characterize a vector with 3 or more entries.

To solve this constrained optimization problem, we'll use (+recall) Lagrange multipliers.

Simple version of Lagrange Multipliers: solve the constrained opt. problem: $\min_{\underline{x}} f(\underline{x})$ subject to $g(\underline{x}) = 0$. $\underline{x} \in \mathbb{R}^N$

Plot this problem in $N=2$ dimensions:



at $\otimes$ (solution):

normal vector to $g(x)=0$ and the contour of f are parallel: $\nabla f = \text{const.}\, \nabla g$. (at $\otimes$)

Lagrange multipliers: solving the constraind problem can be achieved by solving:

3 unknowns (x, λ), 3 eqns. $\begin{cases} \nabla f = \lambda \nabla g, & \lambda \in \mathbb{R} \\ g(x) = 0. \end{cases}$

This procedure can be written as an unconstrained opt. problem: Define $\mathcal{L}(\underline{x}, \lambda) = f(x) + \lambda g(x)$

$$\nabla_x \mathcal{L} = 0 \iff \nabla f + \lambda \nabla g = 0$$

$$\nabla_\lambda \mathcal{L} = 0 \iff g(x) = 0$$

$\mathcal{L}$: "Lagrangian".

Lagrange Multiplies: compute $\overbrace{\text{critical}}^{\text{derivative}=0}$ point of $\mathcal{L}$ as a function of $(\underline{x}, \lambda)$

With $m \geq 1$ constraints: $\mathcal{L}(\underline{x}, \underline{\lambda}) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$, $\underline{\lambda} \in \mathbb{R}^m$

where this corresponds to the opt. problem:

$$\min_{x} f(x) \text{ subject to } \underline{g}(\underline{x}) = \underline{0}$$

$$\underline{g}(\underline{x}) = (g_1(\underline{x}), g_2(\underline{x}), \ldots g_m(\underline{x}))^T$$

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{g}^T \underline{\lambda}$$

Lagrange multipliers: $\nabla_{\underline{x}, \underline{\lambda}} \mathcal{L} = 0$

$$\Downarrow$$

$$\nabla_{\underline{x}} \rightarrow \nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

$\underline{\lambda} \in \mathbb{R}^m$

$f: \mathbb{R}^N \to \mathbb{R}$

$$\nabla_{\underline{\lambda}} \begin{cases} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$$

<u>Remark</u>: Critical points (sol'ns to Lagrange mult) need not be minima of f subject to $g(x) = 0$.

Being a solution of a Lagrange mult problem is a necessary condition ~~for~~ for local minimum, but <u>not</u> sufficient.

A sufficient condition can be formulated involving second derivatives of $\mathcal{L}$.

Back to portfolios:

$$\min_{\underline{w}} \underline{w}^T \underline{\underline{A}} \underline{w} \quad \text{subject to} \quad \langle \underline{w}, \underline{1} \rangle = 1$$
$$\langle \underline{w}, \underline{\mu} \rangle = \mu_p$$

$$f(\underline{w}) = \underline{w}^T \underline{\underline{A}} \underline{w} \qquad g_1(\underline{w}) = \langle \underline{w}, \underline{1} \rangle - 1$$
$$g_2(\underline{w}) = \langle \underline{w}, \underline{\mu} \rangle - \mu_p$$

$$\mathcal{L}(\underline{w}, \underline{\lambda}) = \underbrace{\underline{w}^T \underline{\underline{A}} \underline{w}}_{\sum_{i,j} w_i A_{ij} w_j} + \lambda_1 (\langle \underline{w}, \underline{1} \rangle - 1) + \lambda_2 (\langle \underline{w}, \underline{\mu} \rangle - \mu_p)$$

$$\nabla \mathcal{L} = \underline{0} \implies 2 \underline{\underline{A}} \underline{w} + \lambda_1 \underline{1} + \lambda_2 \underline{\mu} = 0 \quad (1)$$
$$\langle \underline{w}, \underline{1} \rangle = 1 \quad (2)$$
$$\langle \underline{w}, \underline{\mu} \rangle = \mu_p \quad (3)$$

$$(1) \implies \underline{w} = -\tfrac{1}{2} \lambda_1 \underline{\underline{A}}^{-1} \underline{1} - \tfrac{1}{2} \lambda_2 \underline{\underline{A}}^{-1} \underline{\mu}$$

$$(2): \quad \lambda_1 \left[-\frac{1}{2} \underline{1}^T \underline{\underline{A}}^{-1} \underline{1} \right] + \lambda_2 \left[-\frac{1}{2} \underline{1}^T \underline{\underline{A}}^{-1} \underline{\mu} \right] = 1$$

$$(3): \quad \lambda_1 \left[-\frac{1}{2} \underline{\mu}^T \underline{\underline{A}}^{-1} \underline{1} \right] + \lambda_2 \left[-\frac{1}{2} \underline{\mu}^T \underline{\underline{A}}^{-1} \underline{\mu} \right] = \mu_p$$

Define $a = \underline{1}^T \underline{\underline{A}}^{-1} \underline{1}$, $b = \underline{1}^T \underline{\underline{A}}^{-1} \underline{\mu}$, $c = \underline{\mu}^T \underline{\underline{A}}^{-1} \underline{\mu}$

$$(2)+(3): \quad \underbrace{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2\mu_p \end{pmatrix}$$

$$\underline{\lambda} = \underline{\underline{M}}^{-1} (-2) \cdot \begin{pmatrix} 1 \\ \mu_p \end{pmatrix}$$

$$= -2 \underline{\underline{M}}^{-1} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_p \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\underline{\underline{M}}^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

$$\Rightarrow \underline{\lambda} = \frac{-2}{ac-b^2} \left[\begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_p \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \frac{-2}{ac-b^2}\left[\begin{pmatrix} c \\ -b \end{pmatrix} + \mu_p \begin{pmatrix} -b \\ a \end{pmatrix}\right]$$

$$= \frac{2}{ac-b^2}\begin{pmatrix} -c + b\mu_p \\ b - a\mu_p \end{pmatrix}$$

Recall: $\underline{w} = -\frac{1}{2}\lambda_1 \underline{\underline{A}}^{-1}\underline{1} - \frac{1}{2}\lambda_2 \underline{\underline{A}}^{-1}\underline{\mu}$

$$\underline{w} = \frac{-1}{ac-b^2}\left[(-c + b\mu_p)\underline{\underline{A}}^{-1}\underline{1} + (b - a\mu_p)\underline{\underline{A}}^{-1}\underline{\mu}\right]$$

$$= \frac{1}{ac-b^2}\left[c\underline{\underline{A}}^{-1}\underline{1} - b\underline{\underline{A}}^{-1}\underline{\mu}\right]$$

$$+ \frac{\mu_p}{ac-b^2}\left[-b\underline{\underline{A}}^{-1}\underline{1} + a\underline{\underline{A}}^{-1}\underline{\mu}\right]$$

$\underline{w}$ ↗ is a critical pt of $\mathcal{I}$ (and we'll assume this corresponds to the global min of the

Opt. problem)

$$\Longrightarrow \underline{v}_0 = \frac{c\,\underline{\underline{A}}^{-1}\underline{1} - b\,\underline{\underline{A}}^{-1}\underline{\mu}}{ac-b^2}$$

$$\underline{v}_1 = \frac{-b\,\underline{\underline{A}}^{-1}\underline{1} + a\,\underline{\underline{A}}^{-1}\underline{\mu}}{ac-b^2}$$

$$\Longrightarrow \underline{w} = \underline{v}_0 + \mu_p \underline{v}_1$$

Desired (risk-optimal) portfolio.

Q: How does the optimal risk depend on μ_p?

$$\sigma_p^{2\,(\mu_p)} = \underline{w}^T \underline{\underline{A}}\, \underline{w}$$

$$= (\underline{v}_0 + \mu_p \underline{v}_1)^T \underline{\underline{A}} (\underline{v}_0 + \mu_p \underline{v}_1)$$

$$= \mu_p^2 [\underline{v}_1^T \underline{\underline{A}}\, \underline{v}_1] + \mu_p [2 \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1] + \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0$$

$$\frac{\sigma_p^2}{v_1^T \underline{\underline{A}} v_1} = \mu_p^2 + \mu_p \frac{2 v_0^T \underline{\underline{A}} v_1}{v_1^T \underline{\underline{A}} v_1} + \frac{v_0^T A v_0}{v_1^T A v_1}$$

$$= \left(\mu_p + \frac{\underline{v}_0^T \underline{\underline{A}} \underline{v}_1}{\underline{v}_1^T \underline{\underline{A}} \underline{v}_1}\right)^2 - \left(\frac{\underline{v}_0^T \underline{\underline{A}} \underline{v}_1}{\underline{v}_1^T \underline{\underline{A}} \underline{v}_1}\right)^2 + \frac{v_0^T A v_0}{v_1^T A v_1}$$

$$\frac{\sigma_p^2}{(v_1^T A v_1)\left[\frac{v_0^T A v_0}{v_1^T A v_1} - \left(\frac{v_0^T A v_1}{v_1^T A v_1}\right)^2\right]} - \frac{\left(\mu_p + \frac{v_0^T A v_1}{v_1^T A v_1}\right)^2}{\frac{v_0^T A v_0}{v_1^T A v_1} - \left(\frac{v_0^T A v_1}{v_1^T A v_1}\right)^2} = 1$$

$$\frac{\sigma_p^2}{\alpha^2} - \frac{\left(\mu_p + \frac{v_0^T A v_1}{v_1^T A v_1}\right)^2}{\beta^2} = 1$$

(This is the graph of a hyperbola.)

$\alpha^2 > 0$ by Cauchy-Schwarz

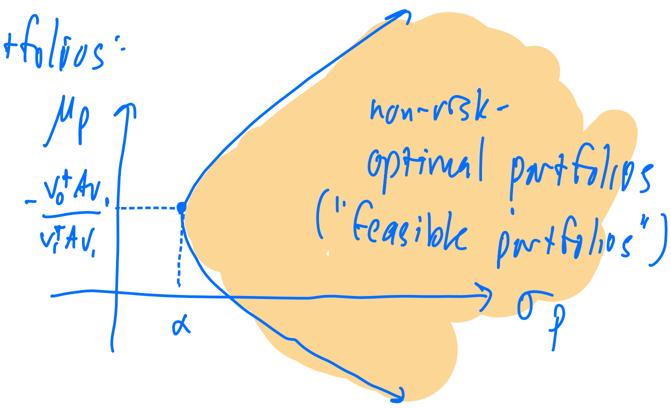
$$\alpha^2 = v_0^T A v_0 - \frac{(v_0^T A v_1)^2}{v_1^T A v_1}$$

$$\beta^2 = \frac{v_0^T A v_0}{v_1^T A v_1} - \left(\frac{v_0^T A v_1}{v_1^T A v_1}\right)^2$$

Vertex: $\mu_p = \frac{-v_0^T A v_1}{v_1^T A v_1}$

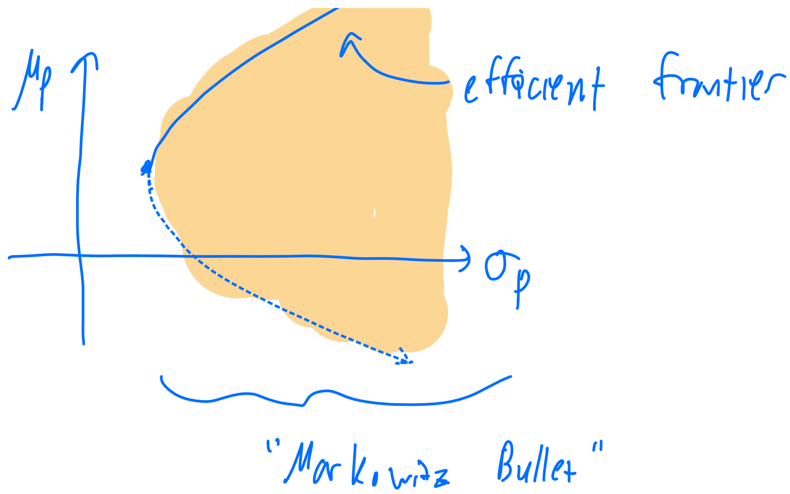
$\sigma_p = \alpha > 0$

Risk-optimal portfolios:



There is a global variance-minimizing portfolio @ the vertex, i.e. at $\mu_p = \frac{-v_0^T A v_1}{v_1^T A v_1} =: \mu_0$

There is an efficient frontier: set of risk-optimal portfolios whose μ_p is maximized for a fixed σ_p.


μ_p
σ_p
efficient frontier
"Markowitz Bullet"

An explicit result

Define some auxilliary quantities:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \mathbf{1}^T \boldsymbol{A}^{-1}\mathbf{1} \\ \mathbf{1}^T \boldsymbol{A}^{-1}\boldsymbol{\mu} \\ \boldsymbol{\mu}^T \boldsymbol{A}^{-1}\boldsymbol{\mu} \end{pmatrix},$$

$$\boldsymbol{v}_0 = \frac{c\boldsymbol{A}^{-1}\mathbf{1} - b\boldsymbol{A}^{-1}\boldsymbol{\mu}}{ac - b^2} \in \mathbb{R}^N, \qquad \boldsymbol{v}_1 = \frac{-b\boldsymbol{A}^{-1}\mathbf{1} + a\boldsymbol{A}^{-1}\boldsymbol{\mu}}{ac - b^2} \in \mathbb{R}^N.$$

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Theorem

Consider the N-security Markowitz portfolio with our assumptions. Then:

- *The unique risk-optimal portfolio with expected return rate μ_P is $\boldsymbol{w} = \boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1$.*
- *The set of risk-optimal portfolios is a parametric curve in the (σ_P, μ_P) plane, given by,*

$$\frac{\sigma_P^2}{\sigma_G^2} + \frac{(\mu_P - \mu_G)^2}{d^2} = 1.$$

- *The global risk-minimizing portfolio has coordinates (σ_G, μ_G) given by*

$$\sigma_G^2 = \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 - \frac{(\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1)^2}{\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1}, \qquad \mu_G = -\frac{\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1}{\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1}.$$

The efficient frontier

As with the 2-security case, the efficient frontier corresponds to the portfolios on the "upper half" of the risk-minimizing hyperbola.

I.e., a risk-minimizing portfolio with coordinates (σ_P, μ_P) is *efficient* if $\mu_P \geqslant \mu_G$.

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This region of feasible portfolios is called the *Markowitz bullet*, with the risk-optimal portfolios forming the bullet boundary, the efficient frontier the upper part of this boundary, and the global risk-minimizing portfolio the tip of the bullet.

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Efficient portfolios can correspond to substantially leveraged portfolios.

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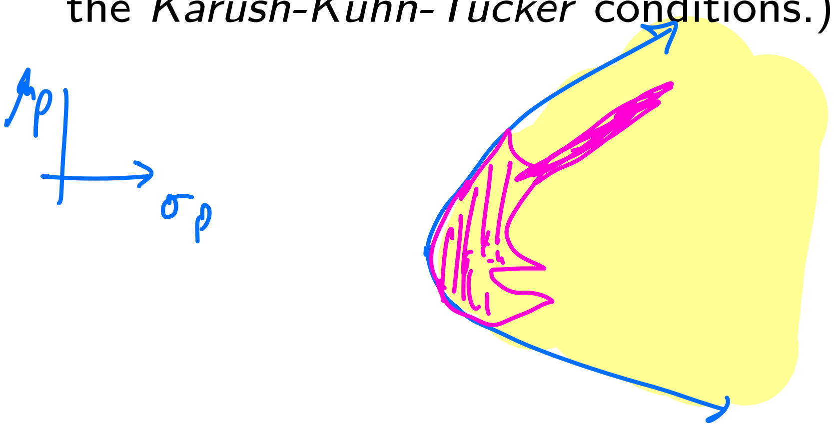
w_i \geqslant 0, \;\; i &= 1, 2, \ldots, N.
\end{aligned}
$$

This is an <u>in</u>equality-constrained optimization problem.

The method of Lagrange multipliers can no longer handle this problem. (We need instead the *Karush-Kuhn-Tucker* conditions.)



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Typically we rely on numerical software to solve this problem, but generically the solution set is some subset of the Markowitz bullet, with the efficient frontier a subset of the bullet boundary.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.