Math 5760/6890: Introduction to Mathematical Finance
The 2-security Markowitz Efficient Frontier

See Petters and Dong 2016, Section 3.2

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September 14, 2023

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A quick recap

For Markowitz 2-security portfolio optimization:

- Return rates $\boldsymbol{R}$ for the two securities are random variables
- Assume first- and second-order statistics of these are available: $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ and $\mathrm{Cov}(\boldsymbol{R})$
- The portfolio is defined by the weights $\boldsymbol{w}$
- The expected return rate of the portfolio is $\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle$
- The squared risk of the portfolio is $\sigma_P^2 = \mathrm{Var}\langle \boldsymbol{R}, \boldsymbol{w} \rangle = \boldsymbol{w}^T \mathrm{Cov}(\boldsymbol{R}) \boldsymbol{w}$

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We typically seek a risk-optimal portfolio for a given target mean μ_P:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \;\; \text{subject to} \;\; \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1, \;\; \text{and} \quad \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

, A = Cov R

Lessons from last time

- We can formulate an optimization that minimizes risk at a given expected return rate.
- There is typically only one feasible portfolio, hence it's optimal.
- The resulting optimized risk can be *much* lower than the individual security risks.
- The optimal-risk portfolio is *not* necessarily efficient: there can be portfolios having the same risk but higher expected return!

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Today: efficient portfolios and the efficient "frontier".

Picking up from an example last time

Example

Consider the problem of constructing a (Markowitz) 2-security portfolio. The individual securities have expected return rates and covariance given by,

$$\mathbb{E}\boldsymbol{R}(T) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \mathrm{Cov}\boldsymbol{R}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We've computed the risk-optimal portfolio as a function of μ_P:

$$\boldsymbol{w} = \begin{pmatrix} \frac{6-\mu_P}{4} \\ \frac{\mu-2}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{\mu_P}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\frac{\mu_P-2}{4}$

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And we've used this to determine the optimal risk σ_P for a fixed μ_P:

$$\sigma_P^2 = \frac{3}{8}\mu_P^2 - 3\mu_P + \frac{13}{2}$$

$$= 3/8\,(\mu_p - 4)^2 + 1/2$$

Optimal portfolios

$$\sigma_P^2 = \frac{3}{8}\mu_P^2 - 3\mu_P + \frac{13}{2} = \frac{3}{8}(\mu_P - 4)^2 + \frac{1}{2}$$

This relationship can be used to identify the set of all possible risk-optimal portfolios as a graph of valid (σ_P, μ_P) pairs.

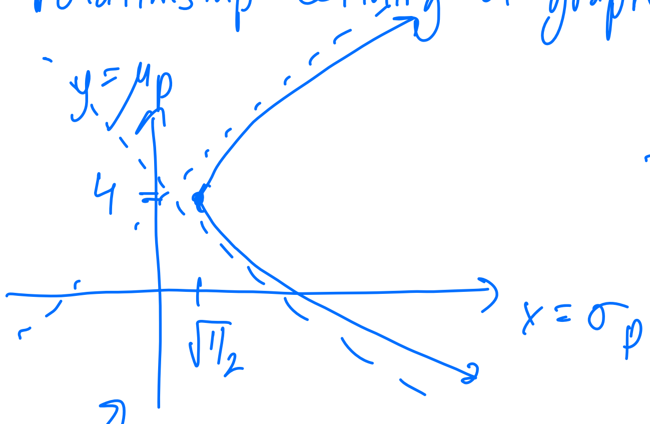
$$\rightarrow \sigma_P^2 - \frac{3}{8}(\mu_P - 4)^2 = \frac{1}{2}$$

$$\frac{\sigma_P^2}{1/2} - \frac{(\mu_P - 4)^2}{\frac{8}{3}\cdot\frac{1}{2}} = 1 \rightarrow \frac{\sigma_P^2}{1/2} - \frac{(\mu_P - 4)^2}{4/3} = 1$$

$$(x, y) = (\sigma_P, \mu_P)$$

$$\frac{x^2}{1/2} - \frac{(y-4)^2}{4/3} = 1$$

→ relationship defining a graph of a hyperbola (!!)



$$\frac{y-4}{4/3} = \pm \frac{x}{\sqrt{1/2}}$$ (slant asymptotes)

Graph of all risk-optimal portfolios.

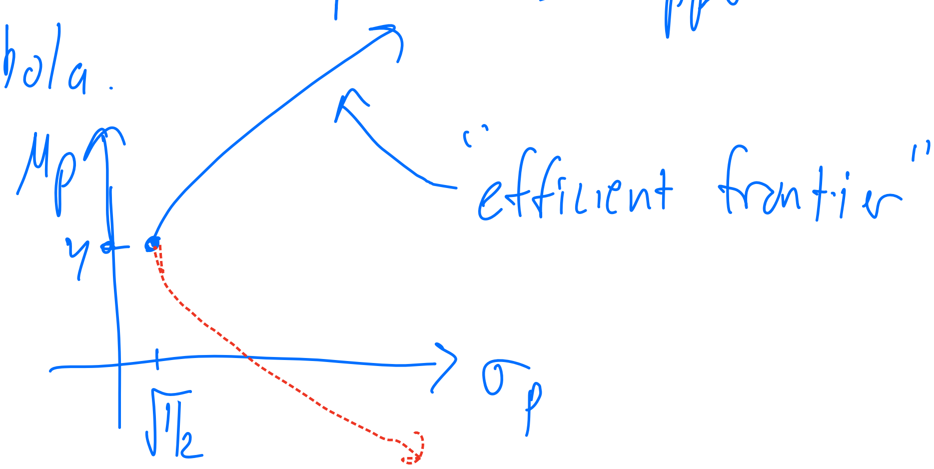
Which ones are efficient?

Non-

Efficient: if decrease σ_p, I can increase μ_p

if I increase μ_p, I can decrease σ_p.

Set of efficient portfolios: upper half of hyperbola.



Efficient frontiers

The particular aspects we've explored in the previous example are generic:

- The set of risk-optimal portfolios is defined by the graph of a hyperbola in the (σ_P, μ_P) plane.
- The upper half of this graph are efficient portfolios – the *efficient frontier*.

Efficient frontiers

$\sigma_p^2 = \frac{3}{8}(\mu_p - 4)^2 + 1/2$ $\mu_p = 4$, $\sigma_p = \sqrt{1/2}$ → global risk-min. portfolio.

The particular aspects we've explored in the previous example are generic:

- The set of risk-optimal portfolios is defined by the graph of a hyperbola in the (σ_P, μ_P) plane.
- The upper half of this graph are efficient portfolios – the *efficient frontier*.
- There is a *global* risk-optimal portfolio, which corresponds to a particular expected return rate.
- In the (general) 2-security model, any feasible portfolio is risk-optimal.[1]

[1] This is not true if $\mu_1 = \mu_2$.

The general 2-security case

The previous aspects hold in the general 2-security case, assuming $\mu_1 = \mu_2$ and a positive-definite covariance.

Example

Consider the general 2-security setup:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R}),$$

where $\boldsymbol{A}$ is positive-definite and $\mu_1 \neq \mu_2$. Show that risk-optimal portfolios are given by the locus of points (σ_P, μ_P) satisfying,

$$\frac{\sigma_P^2}{a^2} - \frac{(\mu_P - \mu_G)^2}{b^2} = 1.$$

where a, b, μ_G are explicit constants: μ_G is the expected return rate of the global risk-optimal portfolio.

Portfolio constraints: $w_1 + w_2 = 1 \longrightarrow \mu_1 w_1 + \mu_1 w_2 = \mu_1 \quad (1)$

$\mu_1 w_1 + \mu_2 w_2 = \mu_P \quad (2)$

$(2)-(1)$: $w_2(\mu_2-\mu_1)=\mu_p-\mu_1$

$$w_2=\frac{\mu_p-\mu_1}{\mu_2-\mu_1} \qquad w_1=1-w_2=\frac{\mu_2-\mu_p}{\mu_2-\mu_1}$$

$$\underline{w}=\begin{pmatrix}w_1\\w_2\end{pmatrix}=\frac{1}{\mu_2-\mu_1}\begin{pmatrix}\mu_2-\mu_p\\\mu_p-\mu_1\end{pmatrix}$$

$$=\underbrace{\frac{1}{\mu_2-\mu_1}\begin{pmatrix}\mu_2\\-\mu_1\end{pmatrix}}_{\underline{v}_0}+\underbrace{\frac{\mu_p}{\mu_2-\mu_1}\begin{pmatrix}-1\\1\end{pmatrix}}_{\mu_p\underline{v}_1}$$

$$\sigma_p^2=\underline{w}^T\underline{\underline{A}}\,\underline{w}=(\underline{v}_0+\mu_p\underline{v}_1)^T\underline{\underline{A}}(\underline{v}_0+\mu_p\underline{v}_1)$$

$$=\underline{v}_0^T\underline{\underline{A}}\,\underline{v}_0+2\mu_p\underline{v}_1^T\underline{\underline{A}}\,\underline{v}_0+\mu_p^2\underline{v}_1^T\underline{\underline{A}}\,\underline{v}_1$$

$$=\underline{v}_1^T\underline{\underline{A}}\,\underline{v}_1\left[\mu_p^2+2\mu_p\frac{v_1^TAv_0}{v_1^TAv_1}\right]+v_0^TAv_0$$

$$=v_1^TAv_1\left[\mu_p+\frac{v_1^TAv_0}{v_1^TAv_1}\right]^2-\frac{(v_1^TAv_0)^2}{v_1^TAv_1}+v_0^TAv_0$$

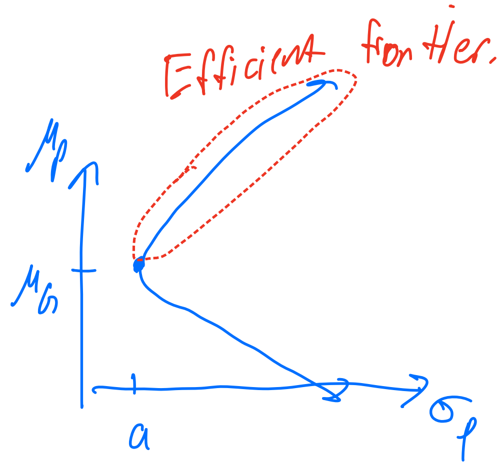
$$\sigma_p^2 - \frac{\left[\mu_p + \frac{v_1^T A v_0}{v_1^T A v_1}\right]^2}{\left(1/v_1^T A v_1\right)} = v_0^T A v_0 - \frac{(v_1^T A v_0)^2}{v_1^T A v_1}$$

$$a^2 = v_0^T A v_0 - \frac{(v_1^T A v_0)^2}{v_1^T A v_1}$$

$$\mu_G = \frac{-v_1^T A v_0}{v_1^T A v_1}$$

$$b^2 = \frac{v_0^T A v_0}{v_1^T A v_1} - \frac{(v_1^T A v_0)^2}{(v_1^T A v_1)^2}$$

$$\Rightarrow \frac{\sigma_p^2}{a^2} - \frac{(\mu_p - \mu_G)^2}{b^2} = 1$$



What about short selling?

For general $\mathbb{E}\boldsymbol{R}$, $\mathrm{Cov}(\boldsymbol{R})$, μ_P, the risk-optimal portfolio could be highly leveraged (even if it's efficient).

In practice, we may want to disallow short selling to avoid such situations.

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The corresponding optimization problem is an augmentation of what we've already seen:

$$\begin{aligned} \min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \text{ subject to } \langle \boldsymbol{w}, \boldsymbol{1} \rangle &= 1, \text{ and} \\ \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle &= \mu_P, \text{ and} \\ w_i \geqslant 0, \quad i &= 1, \ldots, N. \end{aligned}$$

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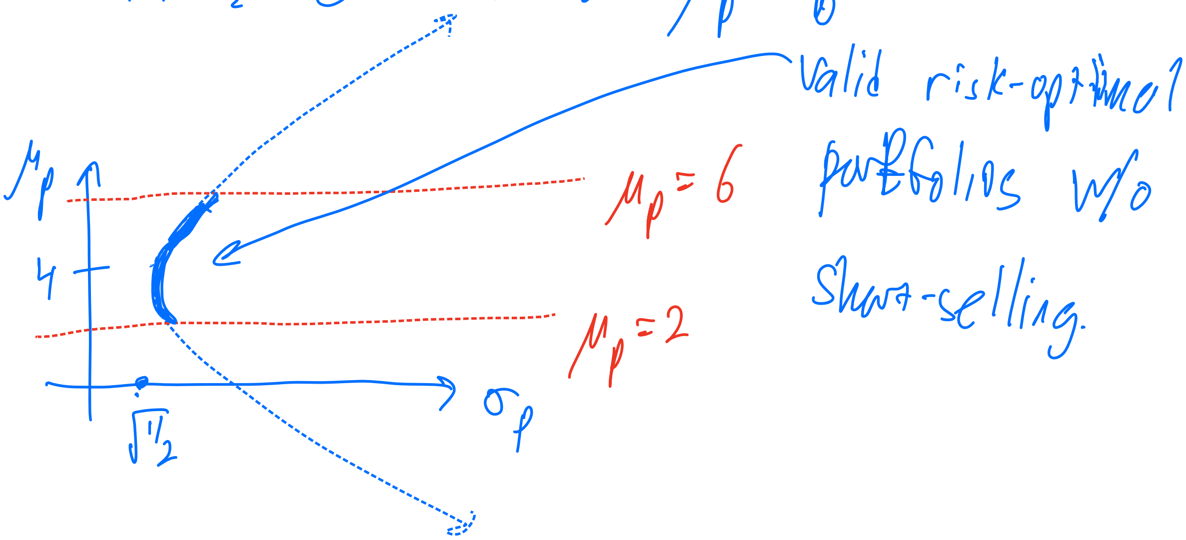
In general, this is a harder optimization problem to solve.

But in the 2-security case, the solution is fairly transparent.

~~Previous~~ First example $\underline{\mu} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ $\underline{\underline{A}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\underline{w} = \begin{pmatrix} \frac{6-\mu_p}{4} \\ \frac{\mu_p - 2}{4} \end{pmatrix}$ $\sigma_p^2 = \frac{3}{8}(\mu_p - 4)^2 + 1/2$

$w_1, w_2 \geq 0 \implies 2 \leq \mu_p \leq 6$



2-security portfolio summary

In the 2-security case:

- Generally, no optimization is needed for 2-security portfolios: the weight constraints uniquely identify portfolios
- Many risk-optimal portfolios are *not* efficient
- The set of risk-optimal portfolios can be explicitly parameterized + plotted
- There is a global variance-minimizing portfolio – investors might not want this portfolio.

Moving to N-security portfolios

Many of the lessons we've learned from 2-security portfolios will hold in the N-security case:

- Risk-optimal portfolios can be computed and plotted
- The efficient frontier is visually identifiable
- There is a globally risk-optimal portfolio

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- The efficient frontier is visually identifiable
- There is a globally risk-optimal portfolio

There are some differences that make things more complicated:

- Optimization is generally required
- There are (many) feasible portfolios that are not risk-optimal
- Explicit pen+paper computations become harder

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.