Math 5760/6890: Introduction to Mathematical Finance
2-Security Porftolios

See Petters and Dong 2016, Sections 3.1-3.2

HW #2 due today

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Office hours on Thurs: 12-1pm

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A brief announcement: AWM Mentoring Program

The Utah chapter of the Association for Women in Mathematics (AWM) invites folks to join it's AWM Mentoring Network:

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The Mentoring Network connects graduate students and postdocs
in the mathematics department with undergraduates majoring, minoring,
or interested in mathematics. The program is intended to support
undergraduates in achieving their academic, research, and job-related
goals and to cultivate a sense of community in mathematics.

After your initial meeting, we hope that you will continue to meet
regularly throughout the school year. Mentor-mentee pairs tend to
meet once per month. You'll also be invited to a kickoff event in
early October where you can meet all the undergraduates, graduate
students, and postdocs involved.
```

You need not identify as a woman to get involved. Signup is here:
`https://forms.gle/yVnC868KYmdS1FhB9`

Please email `uofuAWM@math.utah.edu` for more information.

Portfolios

The first major topic in this course is the risk/reward/pricing assessment of financial portfolios.

A *portfolio* is a combination of (at least two) financial assets (typically securities).

Our goal is to analyze+engineer portfolios under some simplifying assumptions:

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- the future behavior of assets is uncertain or random

The Markowitz model

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Because of the reliance on first- and second-order statistics, Markowitz (or "Modern") portfolio theory is an example of *mean-variance* analysis.

Investors in the one-period model

We will consider the one-period model:

- Today at time $t = 0$ all assets have known, deterministic prices

Investors in the one-period model

book: $0 \to t_0$
$T \to t_f$ | mostly: $T=1$

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We will also assume that investors have an internal understanding of their own risk tolerance as a function of reward.

Our job is to engineer a portfolio achieving, e.g., optimal risk for a fixed given reward.

Efficient portfolios

We have discussed enough to articulate a slightly more quantitative description of our goal.

Let $V = V(t)$ be the (total) value of the portfolio, so that

$$R = R(T) = \frac{V(T) - V(0)}{V(0)},$$

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Definition

A portfolio is an **efficient** portfolio if one *cannot*

- decrease risk without decreasing reward
- increase reward without increasing risk

Notation and setup

We have already discussed seen notation that parameterizes portfolios. Let's review:

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Our eventual goal will be to consider the return rate R_P of the portfolio over the time interval $[0, T]$:

$$V(t) = \langle \boldsymbol{S}(t), \boldsymbol{n} \rangle, \qquad R_P(t) := \frac{V(t) - V(0)}{V(0)} = \left\langle \frac{\boldsymbol{S}(t)}{V_0}, \boldsymbol{n} \right\rangle - 1.$$

↑ $V(0)$

Portfolio weights

In general we allow *short selling*, which corresponds to allowing negative weights/shares in the portfolio.

Example

Suppose we have a 3-security portfolio with initial value $\$1000$, where my 3 securities are:

- Security 1: Microsoft stock, $S_1(0) = \$100$
- Security 2: Coca-cola stock, $S_2(0) = \$50$
- Security 3: Ford Motor Company stock, $S_3(0) = \$150$

Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

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Using $w_1 V(0) = \$500$ in capital, this portfolio holds $n_1 = \frac{w_1 V(0)}{S_1(0)} = 5$ shares of Microsoft.

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Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

This portfolio holds $n_2 = \frac{w_2 V(0)}{S_2(0)} = -6$ shares of Coca-cola.

This means that we *borrow* 6 shares of Coca-cola from person A (like a "loan").

Putatively, we sell these shares back to the market (immediately) to generate $6S_2(0) = \$300$ in captial.

At time $t = T$ in the future, we need to repurchase + return these shares to person A: to expect a profit we hope that $S_2(T) < S_2(0)$.

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Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

This portfolio holds $\frac{w_3 V(0)}{S_3(0)} = \frac{16}{3}$ shares of Ford using $\$800$ in capital.

This capital is comprised of \$500 leftover from initial capital plus \$300 in capital raised by shorting security 2.

This portfolio is *leveraged*. (We have an outstanding debt in Coca-cola stock required to form the portfolio.)

Return rates

We are eventually interested in the return rate $R_P(T)$ of the portfolio. It is notationally useful to rewrite things in terms of return rates for each individual security:

$$R_i(t) := \frac{S_i(t) - S_i(0)}{S_i(0)}, \qquad \boldsymbol{R}(t) = (R_1(t), R_2(t), \ldots, R_N(t))^T .$$

$\underline{R}(t)$ vs. $R_p(t)$

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This definition implies that the full portfolio's return can be written as:

$$\begin{aligned} R_P(T) = \frac{V(T) - V(0)}{V(0)} &= \frac{\langle \boldsymbol{S}(T), \boldsymbol{n}\rangle - V(0)}{V(0)} \\ &= \frac{\langle \boldsymbol{S}(T), \boldsymbol{n}\rangle - \langle \boldsymbol{S}(0), \boldsymbol{n}\rangle}{V(0)} \\ &= \frac{\langle \boldsymbol{S}(T) - \boldsymbol{S}(0), \boldsymbol{n}\rangle}{V(0)} \\ &= \sum_{j=1}^N n_i \frac{S_i(T) - S_i(0)}{V(0)} \\ &= \sum_{j=1}^N \frac{n_i S_i(0)}{V(0)} \frac{S_i(T) - S_i(0)}{S_i(0)} \\ &= \sum_{j=1}^N w_i R_i(T) = \langle \boldsymbol{R}(T), \boldsymbol{w}\rangle. \end{aligned}$$

Return rates, II

We are interested in the return rate $R_P(T)$ of the portfolio.

$$R_P = \langle \boldsymbol{R}, \boldsymbol{w} \rangle, \qquad\qquad R_i(t) := \frac{S_i(t) - S_i(0)}{S_i(0)}.$$

Return rates are dimensionless quantities, and can be represented as a percentage.

Thus, it's easy to compare a portfolio's return rate to, e.g., an alternative risk-free interest rate.

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Under the Markowitz model, we focus on the expected return rate and variance (squared risk) of the return rate:

$$\text{Expected return rate} = \text{“}\mu_P\text{”} = \mathbb{E} R_P,$$

$$\text{Risk} = \text{“}\sigma_P\text{”} = \sqrt{\text{Var}\, R_P}$$

We will engineer the portfolio weights $\boldsymbol{w}$ to optimize these statistics.

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A risk-*optimal* Markowitz portfolio will have the smallest σ_P for a fixed μ_P.

Portfolio optimization

Here is a core problem we will consider: Given

- time-0 asset prices and capital $V(0)$
- time-T distribution of asset prices/return rates
- a target reward (mean time-T portfolio value)

determine a portfolio weight vector that minimizes squared risk.

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In return rates and with math: given statistics about $R_i(T)$ (possibly translated from those of $S_i(T)$) and a target expected return rate μ_P, compute $\boldsymbol{w}$ satisfying,

$$\min_{\boldsymbol{w}} \sigma_P^2 \;\; \text{subject to} \;\; \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1, \;\; \text{and}$$

$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

$\mathbb{E} R_p = \mathbb{E} \langle R, \mu \rangle = \mu_p$

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$\langle a, b \rangle + \langle a, c \rangle = \langle a, b+c \rangle$

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We can make this somewhat more transparent by realizing that σ^2 is a quadratic form of $\boldsymbol{w}$:

$\boldsymbol{\mu} = (\mathbb{E} R_1, \mathbb{E} R_2 \ldots \mathbb{E} R_N)^T$

$$R_P = \langle \boldsymbol{R}, \boldsymbol{w} \rangle \implies \mu_P = \mathbb{E} R_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle,$$

$$\sigma_P^2 = \text{Var}\, R_P = \mathbb{E}\left[R_P - \mu_P\right]^2 = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w},$$

$= \mathbb{E}(\langle \boldsymbol{R}, \boldsymbol{w} \rangle - \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle)^2$

$= \mathbb{E}[\langle \boldsymbol{R} - \boldsymbol{\mu}, \boldsymbol{w} \rangle]^2$

$= \mathbb{E}[\boldsymbol{w}^T (\boldsymbol{R} - \boldsymbol{\mu})^T (\boldsymbol{R} - \boldsymbol{\mu}) \boldsymbol{w}]$

where

$$\boldsymbol{A} = \text{Cov}\, \boldsymbol{R}.$$

$$= \underline{w}^T \, \underbrace{\mathbb{E}\left[(\underline{R} - \underline{\mu})^T (\underline{R} - \underline{\mu}) \right]}_{\mathrm{Cov}(\underline{R}) \;\leftarrow\; \text{"}A\text{"}} \underline{w}$$

$$\mathbb{E}\left[\langle \underbrace{\underline{R} - \underline{\mu}, \underline{w}}_{} \rangle^2 \right]$$

$$\underline{w}^T (\underline{R} - \underline{\mu}) \, \underbrace{\underline{w}^T (\underline{R} - \underline{\mu})}_{(\underline{R} - \underline{\mu})^T \underline{w}}$$

Portfolio optimization, II

Hence, given $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$, and $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$, then the optimization problem is now:

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In the 2-security model, there are only two weights with two linear constraints, so actually there is typically no optimization to be done.

To be consistent with our model assumptions, we'll typically assume that $\boldsymbol{A}$ is positive-definite (no-arbitrage), and we'll allow $\boldsymbol{w}$ to have negative components (short selling).

An example optimization

Example

Consider the problem of constructing a (Markowitz) 2-security portfolio. The individual securities have expected return rates and covariance given by,

$$\underline{\mu} = \mathbb{E}\boldsymbol{R}(T) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \mathrm{Cov}\boldsymbol{R}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For given μ_P, compute the optimal-risk portfolio.

Constraints: $\langle \underline{w}, \underline{1} \rangle = 1 \rightarrow w_1 + w_2 = 1 \quad (1)$

$\langle \underline{w}, \underline{\mu} \rangle = \mu_p \rightarrow 2w_1 + 6w_2 = \mu_p \quad (2)$

$(2) - 2(1) \rightarrow 4w_2 = \mu_p - 2 \Rightarrow w_2 = \frac{\mu_p}{4} - \frac{1}{2}$

$w_1 = 1 - w_2 = 1 - \frac{\mu_p}{4} + \frac{1}{2}$

$= \frac{3}{2} - \mu_p/4$

$$\underline{w} = \begin{pmatrix} 3/2 - \mu p/4 \\ \mu p/4 - 1/2 \end{pmatrix}$$

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What is the squared risk corresponding to this optimal portfolio?

$$\sigma_p^2 = \underline{w}^T \underline{\underline{A}}\, \underline{w}, \quad \underline{\underline{A}} = \mathrm{Cov}(\underline{R}(T))$$

$$\underline{w} = \begin{pmatrix} 3/2 - \mu_p/4 \\ \mu_p/4 - 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{\mu_p}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \underline{V}_0 + \mu_p \underline{V}_1$$

$$V_0 = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \qquad \underline{V}_1 = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\underline{w}^T \underline{\underline{A}}\, \underline{w} = (\underline{v}_0 + \mu_p \underline{v}_1)^T \underline{\underline{A}} (\underline{v}_0 + \mu_p \underline{v}_1), \quad \underline{\underline{A}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0 + 2\mu_p \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1 + \mu_p^2 \underline{v}_1^T \underline{\underline{A}}\, \underline{v}_1$$

$$\underline{\underline{A}}\, \underline{v}_1 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -2-1 \\ 1+2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$\underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1 = \frac{1}{2}(3 \quad -1)\frac{1}{4}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \frac{1}{8}(-9-3) = -3/2$$

$$\underline{v}_1^T \underline{\underline{A}}\, \underline{v}_1 = \frac{1}{4}(-1 \quad 1)^T \frac{1}{4}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \frac{1}{16}(6) = 3/8$$

$$\underline{\underline{A}}\, \underline{v}_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6+1 \\ -3-2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7 \\ -5 \end{pmatrix}$$

$$\underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0 = \frac{1}{2}(3 \quad -1)^T \frac{1}{2}\begin{pmatrix} 7 \\ -5 \end{pmatrix} = \frac{1}{4}(21+5)$$

$$= 13/2$$

$$\sigma_p^2 = \underline{w}^T \underline{\underline{A}}\, \underline{w} = \frac{3}{8}\mu_p^2 - 3\mu_p + 13/2$$

An example optimization

Example

Consider the problem of constructing a (Markowitz) 2-security portfolio. The individual securities have expected return rates and covariance given by,

$$\mathbb{E}\boldsymbol{R}(T) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \mathrm{Cov}\,\boldsymbol{R}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For given μ_P, compute the optimal-risk portfolio. ✓

What is the squared risk corresponding to this optimal portfolio? ✓

Compute the corresponding optimal risks for $\mu_P = 3$, and for $\mu_P = 5$.

$$\sigma_p^2 = 3/8\, \mu_p^2 - 3\mu_p + 13/2$$

$$= 3/8 \left(\mu_p^2 - 8\mu_p\right) + 13/2 = 3/8\left(\mu_p - 4\right)^2 - 16 \cdot \tfrac{3}{8} + 13/2$$

$$= 3/8\left(\mu_p - 4\right)^2 + 1/2$$

$$\sigma_p^2 \Big|_{\mu_p = 3} = \frac{3}{8}(-1)^2 + \frac{1}{2} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$$

$$\sigma_p^2 \Big|_{\mu_p = 5} = \frac{3}{8}(+1)^2 + \frac{1}{2} = \frac{7}{8}$$

- $\underset{\sigma_p^2}{\underset{\shortparallel}{\operatorname{var} R_p}} < \operatorname{var} R_i \quad i = 1, 2 \qquad (\operatorname{Var} R_i = 2)$

- Investor wants a portfolio with $\mu_p = 3$.

  $\Longrightarrow$ investors should prefer $\mu_p = 5$.

- $\mu_p = 3 \rightarrow$ NOT an efficient portfolio

A summary

For Markowitz 2-security portfolio optimization:

- We can formulate an optimization that minimizes risk at a given expected return rate.
- There is typically only one feasible portfolio, hence it's optimal.
- The resulting optimized risk can be *much* lower than the individual security risks.
- The optimal-risk portfolio is *not* necessarily efficient: there can be portfolios having the same risk but higher expected return!

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.