Math 5760/6890: Introduction to Mathematical Finance

See Petters and Dong 2016, Sections 2.1-2.5

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Interest

Interest is the price for access to money (typically someone else's).

Why does (temporary) access money cost more money?

- Money is a resource
- "Opportunity cost": If a lender had not given money, they could have used it to make money in some other way
- Inflation: the purchasing power of \$1 generally decreases with time

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The amount of interest charged to a lendee is typically levied through a *rate* (price per unit time).

Hence, loans almost uniformly specify some basic terms:

- **Principal**: the amount of money the lender temporarily gives to the lendee.
- **Rate**: i.e., the "interest rate", which is the per-time-unit cost that the lendee must bear.

Frequently loans must be paid in full (principal + interest), either in as a lump sum or in installments, by some specified end time of the loan, the **term**.

Interest rates

The time unit for quoting as rate is important. One of the most common time units is years.

Example

A lendee takes out a loan of $500, with an annual (yearly) interest rate of 5%.

This means that at the end of one year, the total interest owed is $0.05 \times 500 = \$25$.

As seen above, the interest rate can be discussed either as a percentage (5%) or a decimal (0.05).

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Hence generally if the interest rate is r (as a decimal) on a principal of amount P, and t units of time have passed, then the total ("simple") interest owed is,

$$\text{"Simple" Interest} = trP.$$

This is just the charge for the service of instituting a loan – the principal P is also a cost the lendee must bear!

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Interest rate conventions and extensions:

- We will always assume that the interest rate r is strictly positive and that $r \ll 1$. (Although $r = 0$ has uses and $r < 0$ is possible.)
- The rate r can depend on time t.
- For simple interest, the rate can easily be converted to other units of time: E.g., 5% annual rate equals a $5/12\% \approx 0.417\%$ monthly rate.

Interest minutiae

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From the lender/investor point of view: assuming an end-term lump sum payment, the total worth of the loan as a financial instrument depends on the time elapsed t:

$$\text{Loan value} = (1 + rt)P. \qquad \text{(for simple interest)}$$

The *total return rate* is the *relative* change of the financial instrument:

$$\text{Total return rate} := \frac{\text{(Loan value) - P}}{P},$$

where we emphasize again that Loan value depends on t.

The *total return amount* is the numerator of the expression above.

A glimpse at present value

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The total bond payout at expiry of the term is $1,060$. $60 = 3 \times \frac{2}{100} \cdot 1000$

The total return rate is 6% (over 3 years!)

Generally I could say that the value $V(t)$ of my investement at time t is

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Bonds are typically sold for a fixed maturity amount (end-term value), at the present value.

I.e., If I sold a bond that will be worth \$1,000 in 3 years time at a rate of 2%, then the price of the bond today should be,

$$V(0) = \frac{V(t)}{1 + rt} = \frac{1000}{1 + 0.02 \times 3} \approx \$943.40$$

The coefficient $(1 + rt)^{-1}$ is called the *discount factor*.

Compound interest

From the lending/investing point of view simple interest has a deficiency: after every period, interest is accrued, but may not yet be paid back, and hence is effectively additional money loaned.

Compound interest addresses this "problem" by levying the interest rate on the principal plus any accrued interest.

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Suppose I invest \$1,000 into a bond, with a(n annual) rate of 2%.
Assume compound interest, compounded every year.

The total value $V(t)$ for integers t is then,

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\begin{aligned}
V(t) &= V(0) + \underbrace{r\overset{V(0)}{\cancel{V_0}}}_{\text{Year 1 interest}} + \underbrace{rV(1)}_{\text{Year 2 interest}} + \underbrace{rV(2)}_{\text{Year 3 interest}} + \cdots \\
&= V(0) + rV(0) + r(V(0) + rV(0)) + r[V(0) + rV(0) + r(V(0) + rV(0))] + \cdots \\
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Or more transparently:

$$V(t) = (1+r)V(t-1) = (1+r)(1+r)V(t-2) = \cdots = (1+r)^t V(0).$$

In general, a ~~loan~~ bond of term 3 years is worth $\$1000 \times (1+0.02)^3 \approx \$1,061.21 > \$1,060$.

Compound interest and periods

Loans involving compound interest typically quote an anuual rate, which is prorated and applied (compounded) across several *periods*. For example, suppose a principal P has a rate of 3% that is compounded weekly (52 weeks per year). Then the total value of the loan after 3 years will be,

$$P\left(1 + \frac{0.03}{52}\right)^{156}.$$

$$= P\left(1 + \frac{0.03}{52}\right)^{3 \cdot 52}$$

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More generally, a loan principal P with (annual) rate r with k-periodic compounding over n periods has value,

$$P\left(1 + \frac{r}{k}\right)^{n}.$$

And *fractional compounding* extends this formula to non-integer periods:

$$\text{Loan value} = P\left(1 + \frac{r}{k}\right)^{x}, \qquad\qquad x \geqslant 0.$$

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Note: many loans apply simple interest (not compound) over fractional periods. We saw in the previous example that compounding interest (annually) was better (for the investor) than simple interest.

Is this true in general?

Simplifying the question (and a smidge of math)

Suppose I have a loan rate r (say an annual rate), and consider an arbitrary number of periods $k \in \mathbb{N}$ for a single time unit (say a year).

The question is: how does compound interest compare to the simple interest?

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(1 year)

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To compare SI versus CI, first recall the Binominal Theorem:

$$(a+b)^k = \sum_{q=0}^{K\,\cancel{n}} a^q b^{k-q} \left(\begin{array}{c} k \\ q \end{array} \right), \qquad \left(\begin{array}{c} k \\ q \end{array} \right) = \frac{k!}{q!(k-q)!}, \qquad 0! = 1$$

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Applying this to CI, we have,

$$\begin{aligned} \frac{CI}{P} = \left(1 + \frac{r}{k}\right)^k &= 1^k \left(\frac{r}{k}\right)^0 \left(\begin{array}{c} k \\ 0 \end{array} \right) + 1^{k-1} \left(\frac{r}{k}\right)^1 \left(\begin{array}{c} k \\ 1 \end{array} \right) + 1^{k-2} \left(\frac{r}{k}\right)^2 \left(\begin{array}{c} k \\ 2 \end{array} \right) + \cdots \\ &= 1 + \frac{r}{k} k + \underbrace{\frac{r^2}{k^2} \frac{k(k-1)}{2} + \ldots}_{\text{Some non-negative stuff}} \\ &\geqslant 1 + r = \frac{SI}{P} \end{aligned}$$

Continuously compounded interest

A k-periodic compounded interest on principal P over t total time units with rate r corresponds to a value of

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$$\lim_{k\uparrow\infty} P\left(1+\frac{r}{k}\right)^{kt} = P\left[\underbrace{\lim_{k\uparrow\infty}\left(1+\frac{r}{k}\right)^{k}}_{e^{r}}\right]^{t} = Pe^{rt}.$$

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Another way to see this is to consider a differential equations model:

$$\text{Change in value per unit time} = (\text{Interest rate}) \times (\text{Current value})$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = rV,$$

and when supplemented with the initial data $V(0) = P$, this yields,

$$V(t) = Pe^{rt}.$$

How do banks compute interest?

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Banks typically compound interest daily, with some nuances:

- Periods are measured with *exact time*, which is the lending time interval in days, minus 1.
- *Exact interest* is computed using 365 days/year
- *Ordinary interest* is computed using 360 days/year (30 days per month for 12 months)

Bank's frequently loan on exact time with ordinary interest.

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Finally, a PSA: the annual percentage rate (APR) r is the interest rate.

It's *not* the simple interest rate you owe.

The annual percentage yield (APY) is the actual yearly percentage change (computed using compounding formulas), e.g., for APR r over 1 year:

$$APY = \left(1 + \frac{r}{360}\right)^{360} - 1 \quad \text{(as a ratio)}$$

Who sets interest rates?

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Long-term interest rates are mostly set by market forces – a particularly powerful influencer is the auctioned price of long-term treasury bonds on the market.

Mortgage rates are determined by market forces, in particular the purchasing and trading of mortgage loans.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.