# Math 5760/6890: Introduction to Mathematical Finance

See Petters and Dong 2016, Sections 1.1-1.3

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

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Office hours: Monday @ 12pm, Thurs @ 11am, WEB 4666


# Mathematical Finance

Mathematical/quantitative finance is the science of determining appropriate prices of financial products.

- Personal banking
  - checking, savings, money market accounts
  - certificates of deposit
  - traveler's checks/money orders
  - mortgages (home/property loans)
  - auto or personal loans
  - debit/credit cards
- Investment banking
  - stocks, bonds, mutual funds
  - options
  - derivative securities

This science is exercised worldwide by banks, traders, governments, companies, researchers, etc.

# Financial markets



Markets are exchanges between those seeking to invest, save, or attract capital.

Typically these savers, investors, and lendees are

- you and me (consumers)
- governments
- private companies

These players seek to buy and sell financial products.

Banks are intermediaries between these players in the market.

There are different types of banks, but in this class we typically won't worry about this distinction.

Loosely speaking, banks pool saved money and use it to make investments and loans.

# Leverage

Consider the following scenario:

- I deposit \$100 in a checking account.
- Using my funds, the bank lends out \$50 to person A for a period of time.[1]
- I ask for all my money back before person A can repay the loan.

The bank is short on funds to accomplish this repayment to me.

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In more technical terms: the bank's **reserve** (cash/capital) is insufficient to meet its **liabilities**.

In this case we say that the banked is **leveraged**. In particular it has used its deposits as leverage to make more money (through a loan).

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The ratio of liabilities (debt) to actual immediately available equity is the **leverage ratio**. Other side of the same coin: the percentage of a deposit that the bank reserves for equity safekeeping is the **reserve ratio**.

In the example above, the bank has a reserve ratio of $50\%$, and a leverage ratio of 2. (Typically written as "2:1".)

In terms of *total* credit+debt generated in this example, my initial \$100 generated \$150 in financial products.

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- Person A buys a thingamabob from person B for \$50.
- Person B deposits the \$50 into a bank (perhaps a different one)
- The bank reserves \$25, and loans out \$25 to person C.
- Person C buys ....

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What is the total credit/debt created by continuing this process indefinitely assuming the same reserve ratio?

- \$100: amount I have credited the bank (the bank's debt)
- \$50: amount the bank has credited person A (person A's debt)
- \$25: amount the second bank has credited person C (person C's debt)
- ...

In total:

$$100 + 50 + 25 + \cdots = 100 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{1/2} = 200$$

Hence, a uniform leverage ratio of R:1 (reserve ratio of $\frac{100}{R}\%$) can generate up to $100R$ in financial products.

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Many countries have non-zero reserve requirements, and reserve requirements are only one regularatory tool to curb financial engineering running amok.

# Interest

An important factor in financial engineering is **interest**, which is the price of credit, that is of holding money or capital (typically someone else's money).

Whenever some entity (the lender) loans money or assets to another entity (the lendee), the lender typically charges interest.

- Interest is *not* a fee associated with the loan transaction.
- Interest is *not* a reward that is tied to growth of equity or assets. (E.g., this is a dividend generated from profit.)

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There are different types of interest rates, and different ways to compute interest – this topic will be our first major consideration in this class.

# Profit and arbitrage

The central goal in finance:

Given enough information about financial products – types of products, prices on different markets, transaction fees, interest, etc. – craft a policy, i.e., a simultaneous/sequential series of actions that increases the sum total of your assets.

This is typically referred to as a profit. Some policies only guarantee a profit after a, perhaps lengthy, period of time.

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A somewhat nuanced strategy for achieving a profit: **arbitrage**.
This is the practice of identifying and implementing a profit-making policy by leveraging a difference in price of an asset.

For example:

- You find the course textbook available from an online bookseller for $100, all expenses included.
- The campus bookstore is offering to buy the course textbook for $200.

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An arbitrage typically exists when there is some kind of inequity in the market: e.g., you have information that others don't have, or you have fewer expenses than others have. There is typically no arbitrage in markets where everyone shares the same information, expenses, and opportunities.

# Risk

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The typical way to model uncertainty in markets is to model outcomes using *probability*. (This will be the focus for the last $\sim$half of the semester.)

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For future reference: in financial math, the more formal definition of an arbitrage is that it is a strategy for which both of the following are true:

- there is zero probability of losing money
- there is positive probability of gaining money

# Financial securities

There are several types of financial securities.

## Example (Stocks)

A **stock** is a partial ownership contract; ownership of stock entitles the bearer to a fractional ownership of an entity (typically a company).

The total stock (security-based ownership) of a company is divided into individual **shares**, which can be purchased.

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- priority access to assets in case the company is liquidated
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- common stock
- preferred stock
- convertible stock
- Class A, B, C shares

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A **bond** is a (type of) loan: the issuer (debtor) is "obliged" to pay back the holder (creditor) a fixed amount after a fixed period of time.

Bond issuers may be companies or entities, but holders of a bond do not enjoy ownership benefits – they are simply lenders.

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The bond yield is derived from the above, typically referring to rate of return of the bond.

Bonds are frequently sold below par and reach par value at maturity.

Some bonds have fixed rates, some have variable (say inflation-adjusted) rates.

"Obliged" means that the issuer is required to pay back the holder with near-ultimate priority. (E.g., bondholders are generally paid before stock holders are paid.)

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## Example (Forward contracts)

A **forward contract** is an agreement between parties to conduct a transaction at a future time with a specified price.

The buyer in this contract is hoping that the asset will increase in value beyond the agreed upon price (a "long position").
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The value of a forward contract to either the buyer or the seller is tied to, or derived from, the performance of an underlying asset.

Hence, this is a type of **derivative** financial instrument.

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## Example (Options)

An **option** is the privilege (but not the obligation) to exercise a transaction at a specified ("strike") price on or before a certain date.

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Typically the holder of an option has purchased the option, paying a "premium", that they hope to recoup.

Two main types of options are:

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A *rational* call option holder typically only exercises the option if the strike price is below market value.
A *rational* put option holder typically only exercises the option if the strike price is above market value.

Options have styles; the "vanilla" ones are American and European options.

# Financial securities

This course studies financial security assets, typically with the goal of understanding

- valuation, either at present or in the future
- risk
- strategies for mitigating risk and maximizing profit

# References I

 Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.