Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 9

Brownian motion

Due: Tuesday, Nov 21, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) Recall that Brownian motion, which we identify with the log-return of an asset price, does not have bounded variation. Here is an exercise to motivate why this is consistent with our finance models: Let T, μ, σ, S_0 be fixed, with $T, \sigma, S_0 > 0$. Consider a real-world CRR model for the price S_n of the security S. Show that in the limit as $n \to \infty$, the *variation* of the log-returns of the real-world CRR trajectory is unbounded:

$$\lim_{n\to\infty} \sum_{j=1}^{n} |\log S_j - \log S_{j-1}| = \infty.$$

(This statement is true with probability 1.)

2.) In this exercise, you will provide some evidence to support the fact that $[B]_T = T$, where $B(t)$ is a standard Brownian motion. Define a discrete-time approximation of the quadratic variation of Brownian motion as,

$$Q_n \coloneqq \sum_{j=1}^{n} (B(t_j) - B(t_{j-1}))^2, \qquad t_j = jh_n, \qquad h_n \coloneqq \frac{T}{n}.$$

Show that Q_n has the following first- and second-order statistics for a fixed n:

$$\mathbb{E}Q_n = T, \qquad \mathrm{Var}Q_n = \frac{2T^2}{n}.$$

(Hence, as $n \to \infty$, Q_n converges to mean-T random variable with variance 0.)
Here is a helpful fact to aid computations: if $X \sim \mathcal{N}(0, 1)$, then $\mathbb{E}X^4 = 3$.