Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2023
Homework 8 Solutions
Brownian motion

Due: Tuesday, Nov 21, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) Recall that Brownian motion, which we identify with the log-return of an asset price, does not have bounded variation. Here is an exercise to motivate why this is consistent with our finance models: Let T, μ, σ, S_0 be fixed, with $T, \sigma, S_0 > 0$. Consider a real-world CRR model for the price S_n of the security S. Show that in the limit as $n \to \infty$, the *variation* of the log-returns of the real-world CRR trajectory is unbounded:

$$\lim_{n\to\infty} \sum_{j=1}^{n} |\log S_j - \log S_{j-1}| = \infty.$$

(This statement is true with probability 1.)

Solution: The CRR binomial tree model is that $L_j = \log S_{j+1} - \log S_j$ is a Bernoulli(p_n) random variable taking values $\log u_n$ with probability p_n and $\log d_n$ with probability $1 - p_n$. The real-world CRR equations prescribe the values for (p_n, u_n, d_n):

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right),$$

where $h_n = T/n$. Note then that,

$$|\log S_j - \log S_{j-1}| = \left|\log \frac{S_j}{S_{j-1}}\right| = |\log L_{j-1}| = \begin{cases} |\log u_n|, & \text{with probability } p_n \\ |\log d_n|, & \text{with probability } 1 - p_n \end{cases}$$

However, the real-world CRR equations imply,

$$|\log u_n| = |\log d_n| = \sigma\sqrt{h_n}.$$

Hence, with probability 1,

$$|\log S_j - \log S_{j-1}| = \sigma\sqrt{h_n},$$

and therefore,

$$\begin{aligned} \lim_{n\to\infty} \sum_{j=1}^{n} |\log S_j - \log S_{j-1}| &= \lim_{n\to\infty} \sum_{j=1}^{n} \sigma\sqrt{h_n} = \sigma \lim_{n\to\infty} n\sqrt{h_n} \\ &= \sigma \lim_{n\to\infty} \sqrt{Tn} = \infty, \end{aligned}$$

as claimed.

2.) In this exercise, you will provide some evidence to support the fact that $[B]_T = T$, where $B(t)$ is a standard Brownian motion. Define a discrete-time approximation of the quadratic variation of Brownian motion as,

$$Q_n \coloneqq \sum_{j=1}^{n} (B(t_j) - B(t_{j-1}))^2, \qquad t_j = jh_n, \qquad h_n \coloneqq \frac{T}{n}.$$

Show that Q_n has the following first- and second-order statistics for a fixed n:

$$\mathbb{E}Q_n = T, \qquad \text{Var}Q_n = \frac{2T^2}{n}.$$

(Hence, as $n \to \infty$, Q_n converges to mean-T random variable with variance 0.)
Here is a helpful fact to aid computations: if $X \sim \mathcal{N}(0,1)$, then $\mathbb{E}X^4 = 3$.
Solution: First, we note that by standard properties of Brownian motion:

$$B(t_j) - B(t_{j-1}) \sim \mathcal{N}(0, h_n),$$

and that

$$X_j \coloneqq B(t_j) - B(t_{j-1}), \qquad \{X_j\}_{j=1}^{n} \text{ are iid.}$$

We will utilize the hint at this time: since $X_j \sim \mathcal{N}(0, h_n)$, then $\frac{1}{\sqrt{h_n}} X_j \sim \mathcal{N}(0,1)$, and so

$$\frac{1}{h_n^2}\mathbb{E}X_j^4 = \mathbb{E}\left(\frac{1}{\sqrt{h_n}} X_j\right)^4 = 3 \implies \mathbb{E}X_j^4 = 3h_n^2.$$

Returning to computing statistics of Q_n, we have:

$$\mathbb{E}Q_n = \sum_{j=1}^{n} \mathbb{E}(B(t_j) - B(t_{j-1}))^2 = \sum_{j=1}^{n} \text{Var}X_j = nh_n = T,$$

which shows the desired first-order statistic equality. The second equality requires us to compute,

$$\text{Var}Q_n = \sum_{j=1}^{n} \text{Var}(B(t_j) - B(t_{j-1}))^2 = \sum_{j=1}^{n} \text{Var}X_j^2.$$

Note that to compute the desired value, we need to compute $\text{Var}X_j^2$, where X_j is a standard normal random variable. This can be directly:

$$\text{Var}X_j^2 = \mathbb{E}\left(X_j^2 - \mathbb{E}X_j^2\right)^2 = \mathbb{E}X_j^4 - (\mathbb{E}X_j^2)^2 = \mathbb{E}X_j^4 - h_n^2 = 3h_n^2 - h_n^2 = 2h_n^2.$$

Putting this back in our equation for $\text{Var}Q_n$, yields,

$$\text{Var}Q_n = \sum_{j=1}^{n} \text{Var}X_j^2 = \sum_{j=1}^{n} 2h_n^2 = 2n\frac{T^2}{n^2} = \frac{2T^2}{n},$$

again as desired.