Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 8

Continuous-time models

Due: Tuesday, Nov 14, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Define $Y := e^X$, which is a lognormal random variable. Show that

$$\mathbb{E}Y = \exp(\mu + \sigma^2/2).$$

2.) Given (μ, σ^2, T, n), suppose that (p_n, u_n, d_n) are set according to the real-world CRR equations. The inter-period log-return for time t_j is given by,

$$L_j = \begin{cases} \log u_n, & \text{with probability } p_n \\ \log d_n, & \text{with probability } 1 - p_n \end{cases}$$

Show that the standardization of L_j, i.e., the random variable,

$$\widetilde{L}_j = \frac{L_j - \mathbb{E}L_j}{\sqrt{\mathrm{Var} L_j}},$$

has distribution,

$$\widetilde{L_j} = \begin{cases} \frac{1-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } p_n \\ \frac{-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } 1 - p_n \end{cases}$$

3.) (**MATH 6890 students only**) The Central Limit Theorem states that if $\{X_j\}_{j=1}^{\infty}$ is a sequence of iid random variables with mean zero and variance $\mathrm{Var} X_j = \sigma^2$, then,

$$\lim_{n \uparrow \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \sim \mathcal{N}(0, \sigma^2).$$

Now for a fixed $n \in \mathbb{N}$, let $\{X_{n,j}\}_{j=1}^{n}$ be a sequence of n iid zero-mean random variables. (But, for example, their second moments may differ as a function of n.) Provide an example to show that it is not necessarily true that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}$$

converges to a normally distributed random variable as $n \uparrow \infty$.