Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 8 Solutions

Continuous-time models

Due: Tuesday, Nov 14, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Define $Y \coloneqq e^X$, which is a lognormal random variable. Show that

$$\mathbb{E}Y = \exp(\mu + \sigma^2/2).$$

<u>**Solution**</u>: We recall that the probability density function (pdf) of X has the form,

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Hence, the expectation of Y is given by,

$$\begin{aligned}
\mathbb{E}Y &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(x) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + x\right) \mathrm{d}x.
\end{aligned}$$

We complete the square for the term under the exponential:

$$\begin{aligned}
-\frac{(x-\mu)^2}{2\sigma^2} + x &= -\frac{1}{2\sigma^2}\left[x^2 - 2\mu x + \mu^2 - 2x\sigma^2\right] \\
&= -\frac{1}{2\sigma^2}\left[x^2 - x\left(2\mu + 2\sigma^2\right) + \mu^2\right] \\
&= -\frac{1}{2\sigma^2}\left[x^2 - x\left(2\mu + 2\sigma^2\right) + (\mu+\sigma^2)^2 - (\mu+\sigma^2)^2 + \mu^2\right] \\
&= -\frac{1}{2\sigma^2}(x - (\mu+\sigma^2))^2 + \frac{1}{2\sigma^2}\left[2\mu\sigma^2 + \sigma^4\right] \\
&= -\frac{1}{2\sigma^2}(x - (\mu+\sigma^2))^2 + \left(\mu + \frac{\sigma^2}{2}\right).
\end{aligned}$$

Using this in the integral for $\mathbb{E}Y$, we obtain,

$$\begin{aligned}
\mathbb{E}Y &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}\right) \exp\left(\mu + \frac{\sigma^2}{2}\right) \mathrm{d}x \\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right) \left[\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}\right) \mathrm{d}x\right] \\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right),
\end{aligned}$$

where the last equality uses the fact that the term in square brackets is the integral of the pdf of a random variable with a distribution $\mathcal{N}(\mu + \sigma^2, \sigma^2)$, and hence is equal to 1.

2.) Given (μ, σ^2, T, n), suppose that (p_n, u_n, d_n) are set according to the real-world CRR equations. The inter-period log-return for time t_j is given by,

$$L_j = \begin{cases} \log u_n, & \text{with probability } p_n \\ \log d_n, & \text{with probability } 1 - p_n \end{cases}$$

Show that the standardization of L_j, i.e., the random variable,

$$\widetilde{L}_j = \frac{L_j - \mathbb{E} L_j}{\sqrt{\mathrm{Var} L_j}},$$

has distribution,

$$\widetilde{L_j} = \begin{cases} \frac{1-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } p_n \\ \frac{-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } 1 - p_n \end{cases}$$

Solution: Although some of these computations are already done in lecture slides S19, we reiterate some of the results here: With $h_n = T/n$, the real-world CRR equations are,

$$u_n = \exp(\sigma\sqrt{h_n}), \qquad d_n = \exp(-\sigma\sqrt{h_n}), \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right).$$

We note that L_j can be written as an affine transformation of a Bernoulli(p_n) random variable:

$$L_j = \log d_n + X \log \frac{u_n}{d_n} = -\log u_n + 2X \log u_n, \qquad X \sim \text{Bernoulli}(p_n).$$

Knowing (or directly computing) that a Bernoulli(p_n) random variable has statistics,

$$\mathbb{E} X = p_n, \qquad \mathrm{Var} X = p_n(1 - p_n),$$

then with the assistance of the real-world CRR equations we directly compute

$$\begin{aligned} \mathbb{E} L_j &= -\log u_n + 2(\mathbb{E} X) \log u_n = (2p_n - 1)\log u_n = \frac{\mu}{\sigma}\sqrt{h_n}\log u_n = \mu h_n, \\ \mathrm{Var} L_j &= 4\left(\log u_n\right)^2 \mathrm{Var} X = 4p_n(1-p_n)(\log u_n)^2 = 4p_n(1-p_n)\sigma^2 h_n. \end{aligned}$$

Hence, $\widetilde{L}_j = (L_j - \mathbb{E} L_j)/\sqrt{\mathrm{Var} L_j}$ has distribution,

$$\widetilde{L}_j = \begin{cases} \frac{\log u_n - \mu h_n}{\sqrt{4 h_n \sigma^2 p_n (1-p_n)}}, & \text{with probability } p_n, \\ \frac{\log d_n - \mu h_n}{\sqrt{4 h_n \sigma^2 p_n (1-p_n)}}, & \text{with probability } 1 - p_n. \end{cases}$$

Note that,

$$\frac{\log u_n - \mu h_n}{2\sigma\sqrt{h_n}} = \frac{\sigma\sqrt{h_n} - \mu h_n}{2\sigma\sqrt{h_n}} = \frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h_n} = 1 - p_n.$$

where we have used the real-world CRR equations twice. Similarly,

$$\frac{\log d_n - \mu h_n}{2\sigma\sqrt{h_n}} = \frac{-\sigma\sqrt{h_n} - \mu h_n}{2\sigma\sqrt{h_n}} = -\frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h_n} = -p_n$$

Using these facts in our previous expression for the distribution of $\widetilde{L}_j$, we obtain,

$$\widetilde{L_j} = \begin{cases} \frac{1-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } p_n \\ \frac{-p_n}{\sqrt{p_n(1-p_n)}}, & \text{with probability } 1-p_n \end{cases}$$

as desired.

3.) (**MATH 6890 students only**) The Central Limit Theorem states that if $\{X_j\}_{j=1}^{\infty}$ is a sequence of iid random variables with mean zero and variance $\text{Var}X_j = \sigma^2$, then,

$$\lim_{n\uparrow\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \sim \mathcal{N}(0, \sigma^2).$$

Now for a fixed $n \in \mathbb{N}$, let $\{X_{n,j}\}_{j=1}^{n}$ be a sequence of n iid zero-mean random variables. (But, for example, their second moments may differ as a function of n.) Provide an example to show that it is not necessarily true that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}$$

converges to a normally distributed random variable as $n \uparrow \infty$.

Solution: There are several examples possible; here is a simple one: For a fixed n, let

$$X_{n,j} = \begin{cases} \sqrt{n}, & \text{with probability } 1/2 \\ -\sqrt{n}, & \text{with probability } 1/2 \end{cases}$$

It is straightforward to verify that $\mathbb{E}X_{n,j} = 0$ and $\text{Var}X_{n,j} = n$. Now fix a very large n. The (standard) Central Limit Theorem would state that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}$$

is approximately distributed like a $\mathcal{N}(\mu, \widetilde{\sigma}^2)$ random variable, with

$$\mu = 0, \qquad\qquad \widetilde{\sigma}^2 = \text{Var}X_{n,j} = n.$$

But we see that $\widetilde{\sigma}^2$ depends on n, and in particular $\lim_{n\uparrow\infty} \widetilde{\sigma}^2 = \infty$. Hence, as $n \uparrow \infty$ the quantity,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j},$$

has variance that diverges, and hence cannot converge to a normally distributed random variable.