Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 6

The Binomial Pricing Model

Due: Tuesday, Oct 31, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) Consider an $n = 100$-period Binomial Pricing Model with $(p, u, d) = (0.35, 1.2, 0.9)$ and an initial value of $S_0 = 100$.

(a) What is the maximal value of S_{100} under this model? The minimial value?

(b) Compute the probability that $S_{100} \geq 100$.

2.) Let $X \sim \text{Bernoulli}(p)$, and let $\{X_i\}_{i=1}^n$ be n iid copies of X. Let $Y = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$. Throughout this problem, let $a > 0$ be a deterministic constant.

(a) Compute $\mathbb{E}a^X$ with $a > 0$ a deterministic constant.

(b) If V and W are two independent random variables, then $\mathbb{E}(VW) = (\mathbb{E}V)(\mathbb{E}W)$. Use this to compute $\mathbb{E}a^Y$.

(c) Compute the variance of a^Y.

(d) Apply these facts to the Binomial Pricing Model with parameters (p, u, d): with $S_n = S_0 e^L$, where $L = \sum_{i=1}^n L_i = \sum_{i=1}^n \log G_i$ is the log-return, show that,

$$\mathbb{E}\frac{S_n}{S_0} = (pu + (1-p)d)^n,$$

$$\text{Var}\frac{S_n}{S_0} = \left(pu^2 + (1-p)d^2\right)^n - (pu + (1-p)d)^{2n}$$

3.) Consider an n-period Binomial Pricing Model for an asset over the time interval $t \in [0, T]$ with $d = 1/u < 1$. (This is a special type of recombination condition.) Suppose that from historical data we compute a T-time expected return rate r for the asset:

$$\mathbb{E}S_n = S_0(1 + r),$$

where r is a deterministic, positive constant $r > 0$.

(a) Show that the expected value of the gross return, $\mathbb{E}\frac{S_n}{S_0}$, is given by $(pu + (1-p)/u)^n$.

(b) Show that in order for S_n under the given binomial pricing model to achieve an expected gross return rate $(1 + r)$, then u must satisfy,

$$u = \frac{1}{2p}\left(e^\mu \pm \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

where $\mu = \frac{\log(1+r)}{n}$.

(c) Show that $e^{2\mu} > 4p(1-p)$, and hence there are always two real values of u above.

(d) Show that if we choose the minus option in formula with $\pm$ above, then $u < 1$, and hence the only valid choice is the plus option.
