Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2023
Homework 6 Solutions
The Binomial Pricing Model

Due: Tuesday, Oct 31, 2023

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1.) Consider an $n = 100$-period Binomial Pricing Model with $(p, u, d) = (0.35, 1.2, 0.9)$ and an initial value of $S_0 = 100$.

(a) What is the maximal value of S_{100} under this model? The minimial value?

(b) Compute the probability that $S_{100} \geq 100$.

<u>**Solution**</u>:

(a) The maximum value of S_{100} corresponds to 100 instances of the inter-period gross return rate being $u = 1.2$. The corresponding value would be,

$$S_{100} = 100(1.2)^{100} = 8.28 \times 10^9$$

Similarly, the minimal value corresponds to 100 instances of $d = 0.9$ as the inter-period gross return:

$$S_{100} = 100(0.9)^{100} = 2.66 \times 10^{-3}$$

(b) Since $100 = S_0$, then the question is identical to determining the probability that $S_n \geq S_0$. Note that the gross return is given by,

$$\frac{S_n}{S_0} = u^Y d^{100-Y},$$

where $Y \sim \text{Binomial}(100, 0.35)$. By direct computation, we find that

$$u^36d^{100-36} < 1, \qquad\qquad u^37d^{100-37} > 1.$$

Therefore,

$$\Pr\left(\frac{S_n}{S_0} \geq 1\right) = \Pr\left(Y \geq 37\right).$$

Hence, we need to compute this probability for a Binomial$(100, 0.35)$ random variable. Since,

$$\Pr(Y = k) = \left(\begin{array}{c} 100 \\ k \end{array}\right) 0.35^k (1 - 0.35)^{100-k},$$

then,

$$\Pr\left(\frac{S_n}{S_0} \geq 1\right) = \sum_{k=37}^{100} \left(\begin{array}{c} 100 \\ k \end{array}\right) 0.35^k (1 - 0.35)^{100-k} \approx 0.373.$$

2.) Let $X \sim \text{Bernoulli}(p)$, and let $\{X_i\}_{i=1}^n$ be n iid copies of X. Let $Y = \sum_{i=1}^n X_i \sim \text{Binomial}(n,p)$. Throughout this problem, let $a > 0$ be a deterministic constant.

(a) Compute $\mathbb{E}a^X$ with $a > 0$ a deterministic constant.

(b) If V and W are two independent random variables, then $\mathbb{E}(VW) = (\mathbb{E}V)(\mathbb{E}W)$. Use this to compute $\mathbb{E}a^Y$.

(c) Compute the variance of a^Y.

(d) Apply these facts to the Binomial Pricing Model with parameters (p, u, d): with $S_n = S_0 e^L$, where $L = \sum_{i=1}^n L_i = \sum_{i=1}^n \log G_i$ is the log-return, show that,

$$\mathbb{E}\frac{S_n}{S_0} = (pu + (1-p)d)^n,$$
$$\text{Var}\frac{S_n}{S_0} = \left(pu^2 + (1-p)d^2\right)^n - (pu + (1-p)d)^{2n}$$

<u>**Solution**</u>:

(a) Since X has distribution,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p, \end{cases}$$

then we can directly compute,

$$\mathbb{E}a^X = pa^1 + (1-p)a^0 = pa + 1 - p.$$

(b) We write a^Y as,

$$a^Y = a^{\sum_{j=1}^n X_j} = \prod_{j=1}^N a^{X_j}.$$

Using the independent property mentioned in the problem, and since the $\{X_j\}_{j=1}^n$ are independent, we have,

$$\mathbb{E}a^Y = \prod_{j=1}^n \mathbb{E}a^{X_j} = \left(\mathbb{E}a^X\right)^n,$$

where the last equality uses the fact that $\{X_j\}_{j=1}^n$ and X are identically distributed. Using this along with part (a), we have:

$$\mathbb{E}a^Y = (pa + 1 - p)^n$$

(c) Letting $\mu = \mathbb{E}a^Y$ (which we computed above), we write the variance as,

$$\begin{aligned} \text{Var } a^Y &= \mathbb{E}\left(a^Y - \mu\right)^2 \\ &= \mathbb{E}\left(a^Y\right)^2 - 2\mathbb{E}\left(\mu a^Y\right) + \mu^2 \\ &= \mathbb{E}\left(a^Y\right)^2 - 2\mu^2 + \mu^2 \\ &= \mathbb{E}(a^2)^Y - \mu^2. \end{aligned}$$

We already have an expression for μ; to compute the expression for the first term, we note that it's the same as $\mathbb{E}a^Y$, but with the replacement $a \leftarrow a^2$. Therefore, we have:

$$\text{Var } a^Y = \left(pa^2 + 1 - p\right)^n - (pa + 1 - p)^{2n}$$

(d) For the Binomial Pricing Model, recall that the log-return L_j for time-step j is,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Hence, L_j can be written as,

$$L_j = \log d + X_j (\log u - \log d) = \log d + X_j \log \frac{u}{d},$$

where $X_j \sim \text{Bernoulli}(p)$. In particular, this implies that

$$\begin{aligned} e^L = e^{\sum_{j=1}^n L_j} &= e^{n \log d + \log \frac{u}{d} \sum_{j=1}^n X_j} \\ &= d^n \left(e^{\log \frac{u}{d}}\right)^{\sum_{j=1}^n X_j} \\ &= d^n \left(\frac{u}{d}\right)^Y \end{aligned}$$

Then using the results from the previous parts with $a = \frac{u}{d}$, along with $\mathbb{E}cW = c\mathbb{E}W$ and $\text{Var}(cW) = c^2 \text{Var}W$ for a constant c, we have:

$$\begin{aligned} \mathbb{E}e^L &= d^n \left(p\frac{u}{d} + 1 - p\right)^n = (pu + (1-p)d)^n \\ \text{Var } e^L &= d^{2n} \left[\left(p\frac{u^2}{d^2} + 1 - p\right)^n - \left(p\frac{u}{d} + 1 - p\right)^{2n}\right] \\ &= \left(pu^2 + (1-p)d^2\right)^n - (pu + (1-p)d)^{2n}, \end{aligned}$$

as desired.

3.) Consider an n-period Binomial Pricing Model for an asset over the time interval $t \in [0, T]$ with $d = 1/u < 1$. (This is a special type of recombination condition.) Suppose that from historical data we compute a T-time expected return rate r for the asset:

$$\mathbb{E}S_n = S_0(1 + r),$$

where r is a deterministic, positive constant $r > 0$.

(a) Show that the expected value of the gross return, $\mathbb{E}\frac{S_n}{S_0}$, is given by $(pu + (1-p)/u)^n$.

(b) Show that in order for S_n under the given binomial pricing model to achieve an expected gross return rate $(1 + r)$, then u must satisfy,

$$u = \frac{1}{2p}\left(e^\mu \pm \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

where $\mu = \frac{\log(1+r)}{n}$.

(c) Show that $e^{2\mu} > 4p(1-p)$, and hence there are always two real values of u above.

(d) Show that if we choose the minus option in formula with $\pm$ above, then $u < 1$, and hence the only valid choice is the plus option.

<u>**Solution**</u>:

(a) We have that

$$\frac{S_n}{S_0} = \prod_{j=1}^{n} G_j,$$

where G_j is a shifted Bernoulli random variable:

$$G_j = \begin{cases} u, & \text{with probability } p, \\ \frac{1}{u}, & \text{with probability } 1-p \end{cases}$$

Hence, taking expectations and noting that the G_j are independent, we have,

$$\mathbb{E}\frac{S_n}{S_0} = \mathbb{E}\left(\prod_{j=1}^{n} G_j\right) = \prod_{j=1}^{n} \mathbb{E}G_j = \prod_{j=1}^{n} (pu + (1-p)/u) = \left(pu + \frac{1-p}{u}\right)^n,$$

as desired.

(b) We have computed the expected gross return rate in the previous part, and this must match $(1+r)$:

$$\left(pu + \frac{1-p}{u}\right)^n = 1 + r,$$

Raising both sides to the $1/n$ power (and using the definition of μ) yields,

$$pu + \frac{1-p}{u} = e^{\mu}.$$

Multiplying by u on both sides yields the quadratic condition:

$$pu^2 - e^{\mu}u + (1-p) = 0.$$

The two roots of this equation are found from the quadratic formula:

$$u = \frac{1}{2p}\left(e^{\mu} \pm \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

which is the desired formaula.

(c) Since $r > 0$, then $\mu = \frac{1}{n}\log(1+r) > 0$. Since μ is positive, this subsequently implies that $e^{2\mu} > 1$. On the other hand, if we consider the function,

$$g(p) = 4p(1-p),$$

over the interval $p \in [0,1]$, then its extrema occur either at the endpoints, $p = 0, 1$, or at critical points, $p = 1/2$. The values of g at these candidate extrema are,

$$g(0) = g(1) = 0, \qquad\qquad g(1/2) = 1,$$

and hence we have,

$$0 \le g(p) \le 1, \qquad\qquad p \in [0,1].$$

Therefore:

$$e^{2\mu} > 1 \geq g(p),$$

i.e.,

$$e^{2\mu} > 4p(1-p),$$

as desired, which implies that our formula for u corresponds to two distinct real values.

(d) If we choose the "minus" option for u, i.e.,

$$u = \frac{1}{2p}\left(e^{\mu} - \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

then we seek to show the inequality,

$$\frac{1}{2p}\left(e^{\mu} - \sqrt{e^{2\mu} - 4p(1-p)}\right) < 1$$

To determine if this inequality is true, our first step is to rearrange it to the equivalent inequality,

$$e^{\mu} - 2p < \sqrt{e^{2\mu} - 4p(1-p)}. \tag{1}$$

Note that naively squaring both sides of the inequality is not a valid operation unless $|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)}$. (I.e., $-2 < 1$ does not imply that $(-2)^2 < 1^2$.). Hence, we must verify,

$$|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)} \iff (e^{\mu} - 2p)^2 < e^{2\mu} - 4p(1-p),$$

where we have used the fact that $e^{2\mu} - 4p(1-p) > 0$ from part (c). By expanding the quadratic term and simplifying, this last inequality is,

$$-4pe^{\mu} + 4p^2 < -4p + 4p^2 \iff e^{\mu} > 1,$$

which we have already established. Hence $|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)}$ is true, and therefore we may square both sides of (1) to obtain the following inequality that is equivalent to our desired one,

$$(e^{\mu} - 2p)^2 < e^{2\mu} - 4p(1-p),$$

but we have already shown that this is equivalent to $e^{\mu} > 1$, which is already established. Therefore, we indeed have $u < 1$ by choosing the "minus" option, and so the "plus" choice is the only valid choice.