Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2023
Homework 5 Solutions
Capital Market Theory

Due: Thursday, Oct 5, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) (Petters & Dong, Problem 4.1) State the definition of the *market portfolio*.

Solution: Several explanations are possible; some options are as follows: The technical definition of the market portfolio is that it is the unique efficient risky Markowitz portfolio that is the optimal choice of risky investment in the presence of a riskless investment with a fixed risk-free rate. The market portfolio is the unique risky Markowitz portfolio lying on the capital market line. Informally, the market portfolio is the most desired portfolio for investors with access to the fixed risk-free rate, since an investor can efficiently and optimally act using only the riskless investment along with the market portfolio.

2.) (Petters & Dong, Problem 4.15, **parts a, b, d only**) Assume a risk-free rate of 1.5%. Answer the questions below using the information in the following table:

Portfolio	A	B	C	D	E	F
Expected Return	3.2%	8.1%	9.8%	5.1%	10.7%	4.8%
Standard Deviation	2.7%	9.9%	13.7%	6.2%	17%	6.1%

(a) Among the portfolios in the table, which one is closest to the market portfolio? Justify your answer.

(b) Plot the (best guess to the) capital market line (CML) based on your answer in part (a).

(d) Suppose we are willing to make an investment only with $\sigma = 6.2\%$. Is a return of 6.5% a realistic expectation for us?

Solution:

(a) For the given risk-free rate, each portfolio above yields a capital asset line. Since the capital asset line corresponding to the (unknown) market portfolio has the largest slope, then the capital asset line among the portfolios above with the largest slope will correspond to the portfolio closest to market portfolio. (It's possible that one of these portfolios is the market portfolio, but we cannot easily determine this from the given information.) For a portfolio with a given risk and expected return pair (σ_P, μ_P), the slope of the capital asset line is the slope corresponding to the line connecting (σ_P, μ_P) with $(0, r)$, where r is the risk-free rate. I.e.,

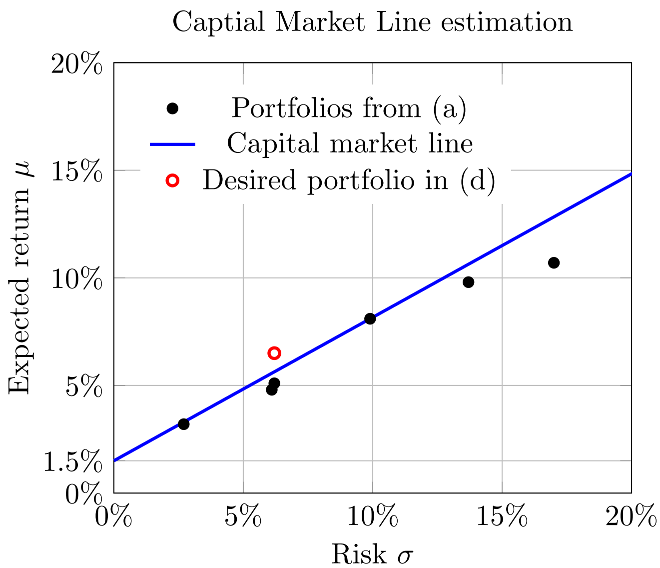
$$\text{slope} = \frac{\mu_P - r}{\sigma_P - 0} = \frac{\mu_P - r}{\sigma_P}.$$

The table above lists (σ_P, μ_P) pairs in each column, so we can directly compute these slopes:

- Portfolio A: slope ≈ 0.630
- Portfolio B: slope ≈ 0.667
- Portfolio C: slope ≈ 0.606
- Portfolio D: slope ≈ 0.581
- Portfolio E: slope ≈ 0.541
- Portfolio F: slope ≈ 0.541

Using the information above, we see that Portfolio B has the largest capital asset line slope, and hence portfolio B is closest to the market portfolio.

(b) The requested plot is shown below, involving the portfolios in part (a) along with the (estimated) captial market line.



(d) Using $(\sigma, \mu) = (6.2\%, 6.5\%)$ in our slope formula yields a slope of approximately 0.806, which is larger than the slope 0.667 of the capital market line determined in the previous parts. Hence, it is *not* realistic for us to expect this (σ, μ) outcome based on the available portfolios. This can also be visualized from the plot in part (b): the point corresponding to the desired portfolio characteristics lies above the capital market line, and so one cannot combine a risky market portfolio with the risk-free asset to achieve these characteristics.

3.) With a risk-free rate of $r > 0$, consider a capital asset line formed by the risk-free asset and a(ny) Markowitz efficient portfolio (σ_P, μ_P). (I.e., the Markowitz portfolio contains only risky securities.)

(a) Write the slope of the capital asset line as a function of μ_P, and maximize this expression to show that the maximum slope of the capital asset line corresponds to an efficient portfolio expected return of

$$\mu_P = -\frac{\alpha_{00} + r\alpha_{01}}{\alpha_{01} + r\alpha_{11}}, \qquad \alpha_{ij} = \boldsymbol{v}_i^T \boldsymbol{A} \boldsymbol{v}_j, \qquad i, j = 0, 1.$$

where $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are vectors corresponding to the solution of the risk-optimal Markowitz portfolio (see slide L09-S05). Hence, this value of μ_P correponds to the market portfolio. (You may assume that critical points correspond to local maxima without proving this.)

(b) Use the expression for the expected return μ_G of the global variance-minimizing portfolio on lecture slide L09-S05 to show that the above expression for the market portfolio mean μ_P is equivalent to,

$$\mu_P = \frac{1}{\mu_G - r}\left(\frac{\alpha_{00}}{\alpha_{11}} - r\mu_G\right),$$

(c) (**Math 6890 students only**) If $\mu_G > r$, show that the market portfolio expected return μ_P given by the formula above satisifes $\mu_P > \mu_G$.

<u>**Solution**</u>:

(a) The captial asset line is formed from the two points $(0, r)$ and (σ_P, μ_P), corresponding to the riskless and a risky security, respectively. Hence, this slope as a function of μ_P is given by,

$$\text{Slope} = f(\mu_P) = \frac{\mu_P - r}{\sigma_P - 0},$$

The risk σ_P corresponding to the risk-optimal Markowitz portfolio is given by,

$$\begin{aligned}\sigma_P^2(\mu_P) &= (\boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1)^T \boldsymbol{A} (\boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1) \\ &= \mu_P^2 \boldsymbol{v}_1 \boldsymbol{A} \boldsymbol{v}_1 + 2\mu_P \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1 + \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 \\ &= \mu_P^2 \alpha_{11} + 2\alpha_{01}\mu_P + \alpha_{00},\end{aligned}$$

where we have used α_{ij} as defined in the problem statement. In order to maximize the slope, we compute critical points of f:

$$f'(\mu_P) = \frac{\sigma_P(\mu_P) - (\mu_P - r)\sigma_P'(\mu_P)}{\sigma_P^2},$$

where via direct calculus we have,

$$\sigma_P'(\mu_P) = \frac{1}{\sigma_P}\left(\mu_P \alpha_{11} + \alpha_{01}\right).$$

Therefore,

$$f'(\mu_P) = \frac{\sigma_P^2 - (\mu_P - r)(\mu_P \alpha_{11} + \alpha_{01})}{\sigma_P^3},$$

which vanishes when the numerator vanishes, yielding the condition,

$$\sigma_P^2 - (\mu_P - r)(\mu_P \alpha_{11} + \alpha_{01}) = 0.$$

Using the expression for $\sigma_P^2(\mu_P)$ above in this condition yields,

$$\mu_P^2(\alpha_{11} - \alpha_{11}) + \mu_P(2\alpha_{01} - \alpha_{01} + r\alpha_{11}) + (\alpha_{00} + r\alpha_{01}) = 0,$$

In other words, μ_P is given by,

$$\mu_P = \frac{-\alpha_{00} - r\alpha_{01}}{\alpha_{01} + r\alpha_{11}}.$$

As stated in the problem, we assume that this critical point corresponds to a local minimum value of $f(\mu_P)$.

(b) We introduce the expected return of the global variance-minimizing Markowitz portfolio:

$$\mu_G = \frac{-\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1}{\boldsymbol{v}_1 \boldsymbol{A} \boldsymbol{v}_1} = -\frac{\alpha_{01}}{\alpha_{11}} \quad \Longrightarrow \quad \alpha_{01} = -\alpha_{11}\mu_G.$$

Using this in our expression for the maximal-slope μ_P yields,

$$\mu_P = \frac{-\alpha_{00} + r\mu_G\alpha_{11}}{-\alpha_{11}\mu_G + r\alpha_{11}} = \frac{-r\mu_G + \frac{\alpha_{00}}{\alpha_{11}}}{\mu_G - r},$$

as desired.

(c) We seek to prove the inequality $\mu_P > \mu_G$. Using the expression from the previous part, this inequality is equivalent to

$$\frac{\frac{\alpha_{00}}{\alpha_{11}} - r\mu_G}{\mu_G - r} > \mu_G.$$

Multiplying the above by $(\mu_G - r)$ along with the assumption $\mu_G - r > 0$ yields,

$$\frac{\alpha_{00}}{\alpha_{11}} - r\mu_G > \mu_G^2 - r\mu_G,$$

i.e.,

$$\mu_G^2 < \frac{\alpha_{00}}{\alpha_{11}}.$$

Using $\mu_G = -\alpha_{01}/\alpha_{11}$, then the above inequality is,

$$\alpha_{01}^2 < \alpha_{00}\alpha_{11},$$

and using the definition of $\alpha_{i,j}$, this is equivalent to:

$$\left|\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_0\right|^2 < \left|\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0\right| \left|\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1\right| \quad \Longleftrightarrow \quad \mu_P > \mu_G. \tag{1}$$

The truth of this inequality is a consequence of the Cauchy-Schwarz inequality. If we define the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{A}}$ as,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{A}} \coloneqq \boldsymbol{u}^T \boldsymbol{A} \boldsymbol{v},$$

then since $\boldsymbol{A}$ is symmetric and positive-definite, we have that this is a valid inner product (it's symmetric, positive-definite, and bilinear). In particular, the expression $\|\boldsymbol{v}\|_{\boldsymbol{A}}^2 \coloneqq \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\boldsymbol{A}}$ is a norm. The inequality (1) is equivalent to,

$$\left|\langle \boldsymbol{v}_0, \boldsymbol{v}_1 \rangle_{\boldsymbol{A}}\right|^2 < \|\boldsymbol{v}_0\|_{\boldsymbol{A}}^2 \, \|\boldsymbol{v}_1\|_{\boldsymbol{A}}^2,$$

which is a strict inequality version of the Cauchy-Schwarz inequality. We can have equality if and only if $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are parallel. This cannot happen since $\mathbf{1}$ and $\boldsymbol{\mu}$ are not parallel vectors. Thus, inequality (1) is true, and therefore the original inequality we sought to prove is established.