Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 4

N-security Markowitz portfolios

Due: Tuesday, Sept 26, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) (Petters & Dong, Problem 3.1) An investor plans to create a portfolio of ten stocks by shorting all of them. Can he use the Markowitz theory as we've introduced it? Explain your answer.

2.) (Petters & Dong, Problem 3.13, *Three Securities*) Suppose that you have \$5,000 to invest in stocks 1, 2, and 3 with current prices

$$\boldsymbol{S}(0) = \begin{pmatrix} \$10.20 \\ \$53.75 \\ \$30.45 \end{pmatrix},$$

along with time-1 expected return vector and covariance matrix given by,

$$\boldsymbol{\mu} = \begin{pmatrix} 0.10 \\ 0.15 \\ 0.075 \end{pmatrix}, \qquad \mathrm{Cov}(\boldsymbol{R}) = \boldsymbol{A} = \begin{pmatrix} 0.03 & -0.04 & 0.02 \\ -0.04 & 0.08 & -0.04 \\ 0.02 & -0.04 & 0.04 \end{pmatrix}.$$

For example, stock 3 has a volatiility of $\sigma_3 = 20\%$ and expected return rate of $\mu_3 = 7.5\%$. Answer the following, using software as appropriate.

(a) Determine the weights needed to create the global minimum-variance portfolio of these three stocks.

(b) Create an efficient portfolio with an expected return rate of 18%. Explicitly state the number of shares one must hold for each stock and how you fund each position. State the portfolio risk and compare it with the maximum risk among the individual stocks.

3.) (N-security global minimizing mean) On slides L09-S05 of the lecture notes, an explicit formula for the mean μ_G of the global variance-minimizing N-security Markowtiz portfolio is provided. Simplify this formula and show that μ_G has the more direct expression:

$$\mu_G = \frac{b}{a} = \frac{\boldsymbol{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{1}^T \boldsymbol{A}^{-1} \boldsymbol{1}},$$

where a, b refers to notation used on slide L09-S05. Use the formula above to justify why the assumption,

$$\boldsymbol{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0,$$

is a reasonable practical assumption to make. (Hint: what does the opposite inequality imply about the global variance-minimizing portfolio?)

4.) (**Math 6890 students only**) (N-security portfolios) Consider the Lagrange multipliers methods for computing the risk-optimal N-security Markowitz portfolio (as done in class and also in the book). With this method, λ_1 corresponds to the constraint $\langle \boldsymbol{w}, \mathbf{1} \rangle = \mathbf{1}$, and λ_2 corresponds to the constraint $\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P$.

(a) Show that if we choose the global variance-minimizing portfolio, then this corresponds to $\lambda_2 = 0$. (The formula $\mu_G = b/a$ from the previous problem can be very helpful here.)

(b) Suppose $\lambda_1 = 0$, and assume $\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0$. Show that the mean of this portfolio is given by,

$$\mu_P = \frac{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \mathbf{1}},$$

and also show that this corresponds to an efficient portfolio. This portfolio is called the *diversified portfolio*. (It may be useful to recall that our general Markowitz portfolio setup assumes that $\mathbf{1}$ and $\boldsymbol{\mu}$ are not parallel vectors.)