Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 4 Solution

N-security Markowitz portfolios

Due: Tuesday, Sept 26, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) (Petters & Dong, Problem 3.1) An investor plans to create a portfolio of ten stocks by shorting all of them. Can he use the Markowitz theory as we've introduced it? Explain your answer.

Solution: We cannot use Markowitz portfolio theory for this. If all 10 stocks are shorted, then the portfolio weight w_i corresponding to the ith stock is negative, for each $i = 1, \ldots, 10$. But Markowitz theory requires $\sum_{i=1}^{10} w_i = 1$. This is impossible if $w_i < 0$ for every i.

2.) (Petters & Dong, Problem 3.13, *Three Securities*) Suppose that you have \$5,000 to invest in stocks 1, 2, and 3 with current prices

$$\boldsymbol{S}(0) = \begin{pmatrix} \$10.20 \\ \$53.75 \\ \$30.45 \end{pmatrix},$$

along with time-1 expected return vector and covariance matrix given by,

$$\boldsymbol{\mu} = \begin{pmatrix} 0.10 \\ 0.15 \\ 0.075 \end{pmatrix}, \qquad \mathrm{Cov}(\boldsymbol{R}) = \boldsymbol{A} = \begin{pmatrix} 0.03 & -0.04 & 0.02 \\ -0.04 & 0.08 & -0.04 \\ 0.02 & -0.04 & 0.04 \end{pmatrix}.$$

For example, stock 3 has a volatiility of $\sigma_3 = 20\%$ and expected return rate of $\mu_3 = 7.5\%$. Answer the following, using software as appropriate.

(a) Determine the weights needed to create the global minimum-variance portfolio of these three stocks.

(b) Create an efficient portfolio with an expected return rate of 18%. Explicitly state the number of shares one must hold for each stock and how you fund each position. State the portfolio risk and compare it with the maximum risk among the individual stocks.

Solution:

(a) The formulas discussed and given in class or in the book (or in later exercises of this assignment) suffice to complete this part, but for completeness we'll repeat the computations for this simplified setup. We seek to solve the optimization problem,

$$\min_{\boldsymbol{w} \in \mathbb{R}^3} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \quad \text{subject to } \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1 \text{ and } \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

The Lagrangian for this problem is,

$$\mathcal{L}(\boldsymbol{w}, \boldsymbol{\lambda}) = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} + \lambda_1 \left(\langle \boldsymbol{w}, \boldsymbol{1} \rangle - 1 \right) + \lambda_2 \left(\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle - \mu_P \right).$$

The critical points of the Lagrangian correspond to the state $(\boldsymbol{w}, \lambda_1, \lambda_2)$ that satisfies,

$$\frac{\partial \mathcal{L}}{\partial(\boldsymbol{w}, \boldsymbol{\lambda})} = 0 \implies \begin{cases} 2\boldsymbol{A}\boldsymbol{w} + \lambda_1 \boldsymbol{1} + \lambda_2 \boldsymbol{\mu} &= 0, \\ \boldsymbol{1}^T \boldsymbol{w} &= 1, \\ \boldsymbol{\mu}^T \boldsymbol{w} &= \mu_P. \end{cases}$$

The first equation above implies that $\boldsymbol{w}$ satisfies,

$$\boldsymbol{w} = -\frac{1}{2}\lambda_1 \boldsymbol{A}^{-1}\boldsymbol{1} - \frac{1}{2}\lambda_2 \boldsymbol{A}^{-1}\boldsymbol{\mu} \tag{1}$$

Using this value of $\boldsymbol{w}$ in the second two equations yields,

$$\lambda_1 \left(-\frac{1}{2}\boldsymbol{1}^T \boldsymbol{A}^{-1}\boldsymbol{1}\right) + \lambda_2 \left(-\frac{1}{2}\boldsymbol{1}^T \boldsymbol{A}^{-1}\boldsymbol{\mu}\right) = 1, \tag{2a}$$

$$\lambda_1 \left(-\frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{A}^{-1}\boldsymbol{1}\right) + \lambda_2 \left(-\frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{A}^{-1}\boldsymbol{\mu}\right) = \mu_P. \tag{2b}$$

Using the values of $\boldsymbol{\mu}$ and $\boldsymbol{A}$ provided, this corresponds to the linear system,

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -175 \\ -19.6875 \\ -2.234 \end{pmatrix}$$

Hence, λ_1 and λ_2 are given by,

$$\begin{aligned} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ \mu_P \end{pmatrix} &= \frac{1}{ac - b^2} \begin{pmatrix} c \\ -b \end{pmatrix} + \frac{\mu_P}{ac - b^2} \begin{pmatrix} -b \\ a \end{pmatrix} \\ &= \begin{pmatrix} -0.6537 \\ 5.76 \end{pmatrix} + \mu_P \begin{pmatrix} 5.76 \\ -51.2 \end{pmatrix} \end{aligned}$$

Hence, the risk-optimal portfolio with expected return μ_P is given by,

$$\begin{aligned} \boldsymbol{w} &= \lambda_1 \left(-\frac{1}{2}\boldsymbol{A}^{-1}\boldsymbol{1}\right) + \lambda_2 \left(-\frac{1}{2}\boldsymbol{A}^{-1}\boldsymbol{\mu}\right) = \lambda_1 \begin{pmatrix} -75 \\ -62.5 \\ -37.5 \end{pmatrix} + \lambda_2 \begin{pmatrix} -8.75 \\ -7.1875 \\ -3.75 \end{pmatrix} \\ &= \begin{pmatrix} -1.3714 \\ -0.5429 \\ 2.914 \end{pmatrix} + \mu_P \begin{pmatrix} 16 \\ 8 \\ -24 \end{pmatrix} \\ &=: \boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1. \end{aligned}$$

Then the squared risk is given by,

$$\begin{aligned} \sigma_P^2 = \boldsymbol{w}^T \boldsymbol{A}\boldsymbol{w} &= \mu_P^2 \left(\boldsymbol{v}_1^T \boldsymbol{A}\boldsymbol{v}_1\right) + \mu_P \left(2\boldsymbol{v}_1^T \boldsymbol{A}\boldsymbol{v}_0\right) + \boldsymbol{v}_0 \boldsymbol{A}\boldsymbol{v}_0 \\ &= 25.6\mu_P^2 - 5.76\mu_P + 0.3269 \end{aligned}$$

From this, and the fact that a polynomial $Ax^2 + Bx + C$ has critical point at $-B/(2A)$, we find that the variance-minimizing value of μ_P is μ_G, given by,

$$\mu_G = \frac{5.76}{51.2} = 0.1125.$$

Therefore, the weights for the variance-minimizing portfolio are,

$$\boldsymbol{w} = \boldsymbol{v}_0 + \mu_G \boldsymbol{v}_1 = \begin{pmatrix} 0.4286 \\ 0.3571 \\ 0.2143 \end{pmatrix}.$$

(b) Create an efficient portfolio with an expected return rate of 18%. Explicitly state the number of shares one must hold for each stock and how you fund each position. State the portfolio risk and compare it with the maximum risk among the individual stocks.

Solution: Note that since $0.18 > \mu_G = 0.1125$, then the risk-optimal portfolio corresponding to $\mu_P = 0.18$ will be efficient. From the previous part, we find that this portolio is given by,

$$\boldsymbol{w}\big|_{\mu_P=0.18} = \left(\begin{pmatrix} -1.3714 \\ -0.5429 \\ 2.914 \end{pmatrix} + \mu_P \begin{pmatrix} 16 \\ 8 \\ -24 \end{pmatrix}\right)\Bigg|_{\mu_P=0.18} = \begin{pmatrix} 1.509 \\ 0.8971 \\ -1.406 \end{pmatrix}$$

With an initial portfolio value of $V(0) = 5000$ and initial per-share prices as given in the problem, this corresponds to the following trading strategy (number of shares held):

$$\boldsymbol{n} = \begin{pmatrix} \frac{w_1 V(0)}{S_1(0)} \\ \frac{w_2 V(0)}{S_2(0)} \\ \frac{w_3 V(0)}{S_3(0)} \end{pmatrix} = \begin{pmatrix} 739.50 \\ 83.46 \\ -230.82 \end{pmatrix},$$

where the negative shares held indicate short selling. The risk for this portfolio is given by,

$$\sigma_P\big|_{\mu_P=0.18} = \sqrt{25.6\mu_P^2 - 5.76\mu_P + 0.3269}\Big|_{\mu_P=0.18} = 0.3457$$

Note that the risk of the individual securities is,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 0.1732 \\ 0.2828 \\ 0.20 \end{pmatrix}$$

Note that σ_P at $\mu_P = 0.18$ is considerably higher than the individual risks of the composite securities; this is sensible since the requested return rate is higher than any individual security return rate, so in order to deliver a higher return than the securities can individually provide, we pay the price of having a higher risk.

3.) (N-security global minimizing mean) On slides L09-S05 of the lecture notes, an explicit formula for the mean μ_G of the global variance-minimizing N-security Markowtiz portfolio is provided. Simplify this formula and show that μ_G has the more direct expression:

$$\mu_G = \frac{b}{a} = \frac{\boldsymbol{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{1}^T \boldsymbol{A}^{-1} \boldsymbol{1}},$$

where a, b refers to notation used on slide L09-S05. Use the formula above to justify why the assumption,

$$\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0,$$

is a reasonable practical assumption to make. (Hint: what does the opposite inequality imply about the global variance-minimizing portfolio?)
Solution: The formula referred to in the question is,

$$\mu_G = -\frac{\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1}{\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1},$$

where $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are vectors that arise during the solution to the Lagrange Multipliers problem. Although the formulas for these vectors already appears on slide L09-S05, we rederive those formulas for completeness: The expressions for $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are computed by solving the Lagrange Multipliers problem for the minimium-risk portfolio weights $\boldsymbol{w}$, which leads to the expression,

$$\boldsymbol{w} = \boldsymbol{v}_0 + \mu_P \boldsymbol{v}_1.$$

We proceed to compute the minimum portfolio weight to identify the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ above. (Again, this is an optional step given slide L09-S05.) The computations in the solution to problem 2 of this assignment sets up this problem (for the 3-security case, but the formulas we use here are the same as in the N-security case), and leads to formula (2) for the 2×2 linear system that determines the Lagrange Multiplers:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \mathbf{1}^T \boldsymbol{A}^{-1} \mathbf{1} \\ \mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} \\ \boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} \end{pmatrix}$$

The exact solution to this 2×2 linear system is given by,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ \mu_P \end{pmatrix} \tag{3}$$

$$= \frac{1}{ac - b^2} \begin{pmatrix} c \\ -b \end{pmatrix} + \frac{\mu_P}{ac - b^2} \begin{pmatrix} -b \\ a \end{pmatrix} \tag{4}$$

The optimal portfolio weights are given by (1):

$$\begin{aligned} \boldsymbol{w} &= \lambda_1 \left(-\frac{1}{2} \boldsymbol{A}^{-1} \mathbf{1} \right) + \lambda_2 \left(-\frac{1}{2} \boldsymbol{A}^{-1} \boldsymbol{\mu} \right) \\ &= \underbrace{\frac{-\frac{c}{2} \boldsymbol{A}^{-1} \mathbf{1} + \frac{b}{2} \boldsymbol{A}^{-1} \boldsymbol{\mu}}{ac - b^2}}_{\boldsymbol{v}_0} + \mu_P \underbrace{\frac{\frac{b}{2} \boldsymbol{A}^{-1} \mathbf{1} - \frac{a}{2} \boldsymbol{A}^{-1} \boldsymbol{\mu}}{ac - b^2}}_{\boldsymbol{v}_1}. \end{aligned}$$

We proceed to compute the terms in our given formula for μ_G:

$$\boldsymbol{A} \boldsymbol{v}_1 = \boldsymbol{A} \frac{\frac{b}{2} \boldsymbol{A}^{-1} \mathbf{1} - \frac{a}{2} \boldsymbol{A}^{-1} \boldsymbol{\mu}}{ac - b^2} = \frac{\frac{b}{2} \mathbf{1} - \frac{a}{2} \boldsymbol{\mu}}{ac - b^2}.$$

Hence, using our formulas for a, b, c, we have:

$$\begin{aligned} \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1 &= \frac{1}{4(ac-b^2)^2} \left(-cb\mathbf{1}^T \boldsymbol{A}^{-1}\mathbf{1} + ca\boldsymbol{\mu}^T \boldsymbol{A}^{-1}\mathbf{1} + b^2 \boldsymbol{\mu}^T \boldsymbol{A}^{-1}\mathbf{1} - ab\boldsymbol{\mu}^T \boldsymbol{A}^{-1}\boldsymbol{\mu} \right) \\ &= \frac{1}{4(ac-b^2)^2} \left(2cba - 2cab - 2b^3 + 2abc \right) \\ &= \frac{2}{4(ac-b^2)^2} \left(acb - b^3 \right) = \frac{b}{2(ac-b^2)} \end{aligned}$$

Similarly,

$$\begin{aligned} \boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1 &= \frac{1}{4(ac-b^2)^2} \left(b^2\mathbf{1}^T \boldsymbol{A}^{-1}\mathbf{1} - ab\mathbf{1}^T \boldsymbol{A}^{-1}\boldsymbol{\mu} - ab\boldsymbol{\mu}^T \boldsymbol{A}^{-1}\mathbf{1} + a^2 \boldsymbol{\mu}^T \boldsymbol{A}^{-1}\boldsymbol{\mu} \right) \\ &= \frac{1}{4(ac-b^2)^2} \left(-2b^2 a + 2ab^2 + 2ab^2 - 2a^2 c \right) \\ &= \frac{2}{4(ac-b^2)^2} \left(ab^2 - a^2 c \right) = \frac{-a}{2(ac-b^2)} \end{aligned}$$

Combining all of this, we have,

$$\mu_G = \frac{-\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1}{\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1} = \frac{-b}{-a} = \frac{b}{a} = \frac{\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\mathbf{1}^T \boldsymbol{A}^{-1} \mathbf{1}},$$

which is what we wished to show. (Note that the definition of a, b, c here differs from that on the slides by a $-\frac{1}{2}$ factor, but this does not affect the formulas for, e.g., μ_G.)

Finally, it is reasonable in practice to assume that $\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0$ since, if not, then $\mu_G \leq 0$, and hence the expected return rate of the global variance-minimizing portfolio is non-positive, which would not be appealing to investors.

4.) (**Math 6890 students only**) (N-security portfolios) Consider the Lagrange multipliers methods for computing the risk-optimal N-security Markowitz portfolio (as done in class and also in the book). With this method, λ_1 corresponds to the constraint $\langle \boldsymbol{w}, \mathbf{1} \rangle = \mathbf{1}$, and λ_2 corresponds to the constraint $\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P$.

(a) Show that if we choose the global variance-minimizing portfolio, then this corresponds to $\lambda_2 = 0$. (The formula $\mu_G = b/a$ from the previous problem can be very helpful here.)

(b) Suppose $\lambda_1 = 0$, and assume $\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0$. Show that the mean of this portfolio is given by,

$$\mu_P = \frac{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \mathbf{1}},$$

and also show that this corresponds to an efficient portfolio. This portfolio is called the *diversified portfolio*. (It may be useful to recall that our general Markowitz portfolio setup assumes that $\mathbf{1}$ and $\boldsymbol{\mu}$ are not parallel vectors.)

<u>**Solution**</u>:

(a) We borrow the derivations given in the solution to problem 3 above: In particular, we have equation (3), which reads,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ \mu_P \end{pmatrix}$$
$$= \frac{1}{ac - b^2} \begin{pmatrix} c \\ -b \end{pmatrix} + \frac{\mu_P}{ac - b^2} \begin{pmatrix} -b \\ a \end{pmatrix}$$

As the problem states, we choose $\mu_P = \mu_G = b/a$. Using this above, we have:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c - \frac{b^2}{a} \\ -b + b \end{pmatrix} = \begin{pmatrix} \frac{1}{a} \\ 0 \end{pmatrix},$$

so that indeed $\lambda_2 = 0$.

(b) Using our formulas for λ_1 and λ_2 above, requiring $\lambda_1 = 0$ yields the expression,

$$\lambda_1 = \frac{1}{ac - b^2} (c - b\mu_P) = 0,$$

which implies that $\mu_P = c/b$. (Note that $b \neq 0$ since we assume $\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \mathbf{1} > 0$.) Using our formulas for a, b, c in the solution to problem 3, we find:

$$\mu_P = \frac{c}{b} = \frac{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \mathbf{1}},$$

as desired.

To show that this is an efficient portfolio, we must show that the expected mean of this portfolio is greater than or equal to that of the global variance-minimizing portfolio. I.e., we must show:

$$\mu_P = \frac{c}{b} > \frac{b}{a} = \mu_G,$$

Note that $b < 0$ since $b = -\frac{1}{2} \mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}$ and we assume $\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0$, and $a < 0$ since $a = -\frac{1}{2} \mathbf{1}^T \boldsymbol{A}^{-1} \mathbf{1}$ and $\boldsymbol{A}$ (therefore also $\boldsymbol{A}^{-1}$) is positive-definite. Hence the desired inequality is,

$$ac > b^2,$$

i.e., we require $ac - b^2 > 0$. We clearly see that at least $ac - b^2 \neq 0$ must be the case as otherwise the 2×2 linear system that determines the Lagrange Multipliers λ_1, λ_2 would not have a solution. (It must have a solution since we assume that $\boldsymbol{\mu}$ is not parallel to $\mathbf{1}$ and so the two linear equality constraints are distinct.) To formally show that the inequality is true, we note that using the definitions of a, b, c, this inequality is equivalent to,

$$\left(\mathbf{1}^T \boldsymbol{A}^{-1} \mathbf{1}\right) \left(\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}\right) - \left(\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}\right)^2 > 0. \tag{5}$$

Let's define the following expression for any two vectors $\boldsymbol{v}, \boldsymbol{w}$:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\boldsymbol{A}^{-1}} \coloneqq \boldsymbol{v}^T \boldsymbol{A}^{-1} \boldsymbol{w}.$$

This is a valid inner product since it satisfies symmetry, bilinearity, and positive definiteness. Hence, in particular,

$$\|\boldsymbol{v}\|_{\boldsymbol{A}^{-1}} \coloneqq \sqrt{\langle \boldsymbol{v}, \boldsymbol{v}\rangle_{\boldsymbol{A}^{-1}}},$$

is a well-defined norm. Using all this notation, we see that our desired inequality (5) can be rewritten as,

$$\|\mathbf{1}\|_{\boldsymbol{A}^{-1}}\|\boldsymbol{\mu}\|_{\boldsymbol{A}^{-1}} - \langle \mathbf{1}, \boldsymbol{\mu}\rangle^2_{\boldsymbol{A}^{-1}} > 0,$$

which can be rewritten as,

$$|\langle \mathbf{1}, \boldsymbol{\mu}\rangle_{\boldsymbol{A}^{-1}}|^2 < \|\mathbf{1}\|_{\boldsymbol{A}^{-1}}\|\boldsymbol{\mu}\|_{\boldsymbol{A}^{-1}}.$$

We recognize this as the Cauchy-Schwarz inequality, which is clearly true with an inequality ($\leq$) sign. However, equality ($=$) can happen if and only if $\mathbf{1}$ and $\boldsymbol{\mu}$ are parallel, which we assume is not the case. Hence, (5) is true (with a strict inequality), so that $\mu_P > \mu_G$ and hence the portfolio considered in this problem is an efficient portfolio.