Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2023
Homework 3 Solution
2-security Markowitz portfolios

Due: Tuesday, Sept 19, 2023

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1.) (Markowitz 2-security portfolios) Consider a 2-security portfolio having per-unit asset prices $S_1(t)$ and $S_2(t)$. Assume the following statistics for these assets:

$$\begin{aligned} &S_1(0) = 100 \text{ (with probability 1)} && \mathbb{E}S_1(1) = 110, && \sqrt{\mathrm{Var}\, S_1(1)} = 20 \\ &S_2(0) = 50 \text{ (with probability 1)} && \mathbb{E}S_2(1) = 75, && \sqrt{\mathrm{Var}\, S_2(1)} = 40, \end{aligned}$$

along with $\mathrm{Cov}(S_1(1), S_2(1)) = -500$.

(a) Show that the return rates $\boldsymbol{R}$ of the individual securities in this setup have statistics,

$$\mathbb{E}\boldsymbol{R}(1) = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix}, \qquad \mathrm{Cov}\boldsymbol{R}(1) = \begin{pmatrix} 0.04 & -0.1 \\ -0.1 & 0.64 \end{pmatrix}$$

(b) Compute the minimum-risk portfolio for a general expected return rate μ_P.

(c) On an expected return rate vs. risk figure, plot the set of optimal (minimum-risk) portfolios and identify the efficient frontier.

(d) An investor seeks to utilize this optimized portfolio corresponding to the expected return rate of $\mu_P = 15\%$. Would you recommend the corresponding portfolio to this person?

Solution:

(a) We translate the given statistics into corresponding statistics for return rates. We know that such portfolios have the return,

$$R(t) = \langle \boldsymbol{w}, \boldsymbol{R} \rangle, \qquad R_i(t) = \frac{S_i(t) - S_i(0)}{S_i(0)},$$

for $i = 1, 2$, where $\boldsymbol{w}$ contains the unknown portfolio weights. Using standard properties of first- and second-order statistics, we have:

$$\begin{aligned} \mu_1 = \mathbb{E}R_1(1) = \frac{\mathbb{E}S_1(1)}{S_1(0)} - 1 = 0.1, &\qquad \sigma_1 = \sqrt{\mathrm{Var}R_1(1)} = \frac{\sqrt{\mathrm{Var}S_1(1)}}{S_1(0)} = 0.2, \\ \mu_2 = \mathbb{E}R_2(1) = \frac{\mathbb{E}S_2(1)}{S_2(0)} - 1 = 0.5, &\qquad \sigma_2 = \sqrt{\mathrm{Var}R_2(1)} = \frac{\sqrt{\mathrm{Var}S_2(1)}}{S_2(0)} = 0.8 \end{aligned}$$

The covariance satisfies similar properties:

$$\mathrm{Cov}(R_1(1), R_2(1)) = \mathrm{Cov}\left(\frac{S_1(1)}{S_1(0)}, \frac{S_2(1)}{S_2(0)}\right) = \frac{1}{S_1(0)S_2(0)}\mathrm{Cov}(S_1(1), S_2(1)) = -0.1.$$

Hence, we have

$$\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}(1) = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix}, \qquad \boldsymbol{A} = \text{Cov}(\boldsymbol{R}(1)) = \begin{pmatrix} 0.2^2 & -0.1 \\ -0.1 & 0.8^2 \end{pmatrix},$$

as desired.

(b) The Markowitz portfolio optimization for this setup is,

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \quad \text{subject to} \quad \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1 \text{ and} \quad \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P$$

Since this is a 2-security portfolio, the two linear constraints will determine $\boldsymbol{w}$ without optimization:

$$\left.\begin{aligned} w_1 + w_2 &= 1 \\ 0.1 w_1 + 0.5 w_2 &= \mu_P \end{aligned}\right\} \Longrightarrow \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 5 - 10\mu_P \\ 10\mu_P - 1 \end{pmatrix} = \frac{1}{4} \left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \mu_P \begin{pmatrix} -10 \\ 10 \end{pmatrix} \right]$$

Hence, given μ_P, the weights above prescribe the optimal (minimal) risk portfolio.

(c) To identify the risk σ associated with the optimal portfolio, we compute the variance of the optimal portfolio:

$$\sigma_P^2 = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = 5.5\mu_P^2 - 1.8\mu_P + 0.165$$

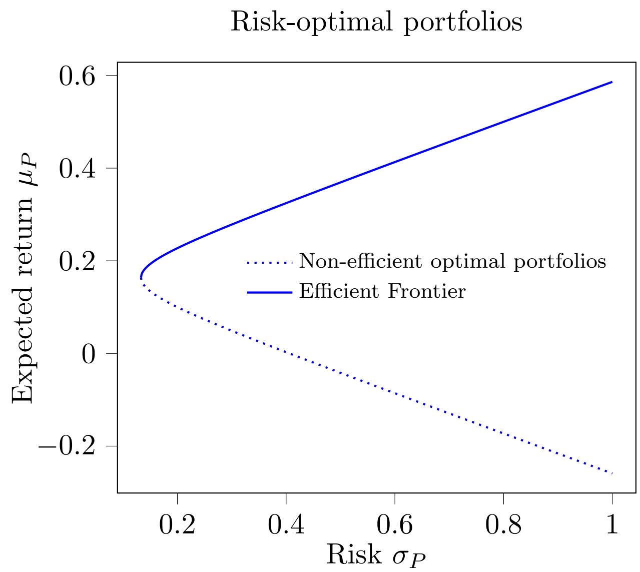
Rearranging this equation yields,

$$\sigma_P^2 - \left(\sqrt{\frac{11}{2}} \mu_P - 0.9 \sqrt{\frac{2}{11}} \right)^2 = 0.165 - 0.81 \frac{2}{11} = \frac{39}{2200},$$

which in turn can be written as,

$$\frac{\sigma^2}{a^2} - \frac{(\mu_P - \mu_0)^2}{b^2} = 1, \qquad (a, b, \mu_0) = \left(\frac{1}{10} \sqrt{\frac{39}{22}}, \ \frac{\sqrt{39}}{110}, \ \frac{9}{55} \right).$$

Hence, this is the graph of a hyperbola with vertex at $(\sigma, \mu_P) = (a, \mu_0)$ opening up toward $\sigma = \infty$. The graph of the optimal portfolios is given below; the efficient frontier is the upper half of this graph, corresponding to optimal expected return rate for a given risk.



(d) Our graph of the efficient frontier in the previous part reveals that $\mu_P = 0.15$ does *not* correspond to an efficient portfolio. In particular, the portfolio at $\mu_P = 0.15$ is on the lower half of the hyperbola. An alternative efficient portfolio corresponds to a portfolio mean located at a reflection of 0.15 around $\mu_0 = 9/55$. (I.e., the symmetric point located on the efficient frontier portion of the hyperbola.) Hence, an investor should prefer a portfolio having expected return,

$$\mu_P = \mu_0 + (\mu_0 - 0.15) \approx 0.177 > 0.15,$$

because the corresponding optimal portfolio has the same risk σ_P as the 15% expected return portfolio, but has higher expected return.

2.) (Arbitrage in portfolios) Consider a 2-security portfolio consisting of asset 1 and asset 2. Assume the time-1 asset return rates R_1 and R_2 have mean and standard deviation (μ_1, σ_1) and (μ_2, σ_2), respectively. Assume that $\sigma_1 + \sigma_2 > 0$, i.e., that at least one security is random.

(a) Recall that the Pearson correlation coefficient between R_1 and R_2 is defined as $\rho := \mathrm{Cov}(R_1, R_2)/(\sigma_1\sigma_2)$. If $\rho = -1$, explicitly construct a zero-risk portfolio using a non-trivial linear combination of assets 1 and 2.

(b) Using the previous result, give a necessary and sufficient condition involving the statistics above that ensures that an arbitrage, i.e., a riskless and (strictly positive) profit strategy, exists.

(c) **(Math 6890 students only)** Extend part of this to the N-security case: Show that if the covariance matrix of the individual security return rates is <u>not</u> positive-definite, instead only of rank $N - 1$, then a riskless security can be constructed, and provide (perhaps opaque but symbolically explicit) conditions on the security statistics that ensure that this riskless security can be used for arbitrage. Your conditions may involve eigenvalues/vectors of the covariance matrix.

<u>**Solution**</u>:

(a) If $\rho = -1$, then the covariance matrix of $\boldsymbol{R}$ can be written as,

$$\boldsymbol{A} \coloneqq \mathrm{Cov}(\boldsymbol{R}) = \begin{pmatrix} \sigma_1^2 & \mathrm{Cov}(R_1, R_2) \\ \mathrm{Cov}(R_1, R_2) & \sigma_2^2 \end{pmatrix} \overset{\rho=-1}{=} \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

The risk of the portfolio is $\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w}$; since $\boldsymbol{A}$ is symmetric, then $\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = 0$ implies that $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}$, i.e., that $\boldsymbol{w}$ is a vector in the nullspace of $\boldsymbol{A}$. The explicit form of $\boldsymbol{A}$ above shows that one such vector is given by,

$$\boldsymbol{w} = \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix}.$$

To make this vector a valid portfolio weight vector, we normalize appropriately:

$$\boldsymbol{w} = \frac{1}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix},$$

whose components sum to unity (making it a valid portfolio weight) and is well-defined since $\sigma_1 + \sigma_2 > 0$. Hence, this forms a zero-risk portfolio.

(b) In order for the portfolio identified in the previous part to be an arbitrage, its mean must be strictly positive. The mean of the portfolio above is,

$$\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle = \frac{\sigma_2\mu_1 + \sigma_1\mu_2}{\sigma_1 + \sigma_2}.$$

Since the denominator is positive, the number above is positive if and only if

$$\sigma_2\mu_1 + \sigma_1\mu_2 > 0.$$

(c) In the N-security case, if $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$ is *not* positive-definite, then there exists some non-zero vector $\boldsymbol{v}$ such that,

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0},$$

so that $\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} = 0$. Note that since $\mathrm{rank}(\boldsymbol{A}) = N - 1$, then $\boldsymbol{v}$ is unique up to a consatnt. In order to be able to normalize $\boldsymbol{v}$ so that it's a valid portfolio weight vector, we must have,

$$\langle \boldsymbol{v}, \boldsymbol{1} \rangle \neq 0.$$

Assuming this, then

$$\boldsymbol{w} \coloneqq \frac{1}{\langle \boldsymbol{v}, \boldsymbol{1} \rangle} \boldsymbol{v},$$

is a valid portfolio weight vector corresponding to a riskless portfolio. This is the only such portfolio since $\boldsymbol{v}$ is unique up to a constant. In order for it to be an arbitrage, it must have positive mean:

$$\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle > 0,$$

where $\boldsymbol{\mu}$ is the mean of the individual securities. Using the expressions we've derived above, this is equivalent to:

$$\frac{\langle \boldsymbol{v}, \boldsymbol{\mu} \rangle}{\langle \boldsymbol{v}, \boldsymbol{1} \rangle} > 0, \qquad \qquad \langle \boldsymbol{v}, \boldsymbol{1} \rangle \neq 0$$

These are conditions that explicitly involve $\boldsymbol{\mu}$, and implicitly involve entries of the second-order statistics $\mathrm{Cov}(\boldsymbol{R})$ since $\boldsymbol{v}$ is an eigenvector of $\mathrm{Cov}(\boldsymbol{R})$.

3.) Consider a Markowitz 2-security portfolio with a given terminal time positive-definite covariance $\mathrm{Cov}(\boldsymbol{R})$ and terminal time mean $\boldsymbol{\mu}$. Assume that,

$$\mu_1 = \mu_2.$$

(a) Show that any Markowitz portfolio must have expected return μ_P given by $\mu_P = \mu_1 = \mu_2$.

(b) For the covariance matrix,

$$\mathrm{Cov}\,(\boldsymbol{R}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

compute both the optimal Markowitz portfolio and its corresponding risk.

Solution:

(a) For notational simplicity, we'll let $\mu = \mu_1 = \mu_2$. The weights of the portfolio must satisfy,

$$\begin{aligned} w_1 + w_2 &= 1 \\ \mu w_1 + \mu w_2 &= \mu_P. \end{aligned}$$

If $\mu = 0$, then clearly $\mu_P = 0 = \mu$. If $\mu \neq 0$, then the second equation is equivalent to,

$$w_1 + w_2 = \frac{\mu_P}{\mu}.$$

In order for this to be consistent with the first equation, we must have $\mu_P = \mu$. (Otherwise the weight constraints are inconsistent and no valid portfolio exists.) Hence, no matter what value of μ, we must have $\mu_P = \mu$.

(b) With our $\mu = \mu_1 = \mu_2$ setup, then the two linear constraints on a Markowitz portfolio are simply the single condition,

$$w_1 + w_2 = 1.$$

Hence, the squared risk of the portfolio with $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$ is,

$$\sigma_P^2 = \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix} = \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 + \left(2\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1\right) w_1 + \left(\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1\right) w_1^2,$$

where

$$\boldsymbol{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We directly compute:

$$\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 = 2, \qquad 2\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1 = -6, \qquad \boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1 = 6,$$

so the risk squared reads,

$$\sigma_P^2 = 6w_1^2 - 6w_1 + 2.$$

Using univariate calculus to compute critical points, we conclude that the minimum of σ_P^2 occurs when,

$$w_1 = \frac{1}{2} \implies \boldsymbol{w} = \left(\begin{array}{c} 0.5 \\ 0.5 \end{array} \right).$$

The (minimal) squared risk for this value of $\boldsymbol{w}$ is,

$$\sigma_P^2 \big|_{w_1 = 0.5} = \frac{1}{2}.$$

Hence, the minimal squared risk is $\frac{1}{\sqrt{2}}$.