Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2023

Homework 1 solutions

Simple valuations

Due: Tuesday, Sept 5, 2023

Submit your homework assignment on Canvas via Gradescope.

1.) (Loan valuation) Suppose the (annual) interest rate for a loan is currently 3% for a loan term of 3 years, and the interest will compound monthly. You estimate that you will be able to afford a maximum monthly payment of \$300 over the loan term. What is the maximum loan amount you can take out and still afford the monthly payments?

Solution: We model this loan as an annuity with the following parameters:

- Interest rate: $r = 0.03$ per year
- Compounded monthly: $k = 12$ terms per year
- $n = 3\text{years} \times 12 = 36$ periods (months)
- $P = 300$ payment every period

The unknown is the loan principal today (the present value); hence our goal is to compute the present value of this annuity. We recall that an n-period annuity with per-period payment of P with a per-period interest rate of r/k has present value:

$$PV = \sum_{j=1}^{36} \frac{P}{(1+r/k)^n} = \frac{Pk}{r}\left(1 - \frac{1}{(1+\frac{r}{k})^n}\right).$$

Using the parameters above in this formula yields a present value of,

$$PV \approx \$10,315.94,$$

which is the maximum affordable loan principal.

2.) (Bond valuation) Bonds are priced based on today's present value of the instrument; the "face value" of a bond is the amount paid to the bearer at expiry of the bond's term ("maturity"). In the simplest setting, the price is based on the sum of two things: (i) the present value of the face value of the bond (which is received at maturity), (ii) the present value of an annuity ("coupon payment") paid to the holder at regular intervals; the payment per period equals the face value times the "coupon/interest rate" (prorated from a quoted annual rate). The "yield to maturity" is an assumed (annual) interest rate on which the present value is discounted from future value.

(a) Determine the price of the following bond: A bond with a face value of \$1000 with a maturity term of 2 years and a coupon rate of 4%. The coupon payments are made semiannually (i.e., twice a year at \$20 per payment) until the bond matures. Throughout, assume a discount rate ("yield to maturity") of 3%.

(b) "Par" refers to a bond face value. Is the bond in part (a) priced below, at, or above par?

(c) (**6890 students only**) Prove in general that if the yield to maturity equals the coupon rate, then the present value of the bond is exactly par.

Solution:

(a) We compute the sum of the present value of the bond face value, plus the present value of the 4 coupon payments made over 2 years:

$$\begin{aligned} PV &= \frac{1000}{\left(1 + \frac{0.03}{2}\right)^4} + \sum_{j=1}^{4} \frac{20}{\left(1 + \frac{0.03}{2}\right)^j} \\ &\approx 942.18 + 77.09 = \$1019.27, \end{aligned}$$

which corresponds to the bond price.

(b) Since the bond price computed above is larger than the face value (\$1000), then this bond is priced above par.

(c) Assume a bond with face value F, an annual coupon rate r, with k coupon payments per year. As stated, we assume the annual discount rate (yield to maturity) is also r. Let the bond term be n periods (i.e., n/k years). The per-period coupon payment is $F\frac{r}{k}$. We seek to show that the present value of this bond is exactly the face value F. The present value of such a bond is given by,

$$\begin{aligned} PV &= \frac{F}{\left(1 + \frac{r}{k}\right)^n} + \sum_{j=1}^{n} \frac{F\frac{r}{k}}{\left(1 + \frac{r}{k}\right)^j} \\ &\overset{q=r/k}{=} \frac{F}{(1+q)^n} + \sum_{j=1}^{n} \frac{Fq}{(1+q)^j} \\ &= \frac{F}{(1+q)^n} + Fq \sum_{j=1}^{n} \frac{1}{(1+q)^j} \\ &= \frac{F}{(1+q)^n} + Fq \left[\frac{1 - \frac{1}{(1+q)^{n+1}}}{1 - \frac{1}{1+q}} - 1 \right] \\ &= \frac{F}{(1+q)^n} + Fq \left[\frac{1 + q - \frac{1}{(1+q)^n}}{1 + q - 1} - \frac{q}{q} \right] \\ &= \frac{F}{(1+q)^n} + F \left[1 - \frac{1}{(1+q)^n} \right] = F, \end{aligned}$$

which proves the result.

3.) (Arbitrage) Thingamabobs currently sell for \$50 each. Today there is an opportunity to enter into a forward contract where a buyer agrees to purchase x thingamabobs ($x \geq 1$, x an integer) from you exactly 1 year from now. The buyer will purchase the first one for \$100, but subsequent ones will be discounted: The cumulative delivery price K for x thingamabobs is

$$K = 100x \left(\frac{9}{10}\right)^{(x-1)}.$$

Your bank has agreed to extend you a loan of up to \$500 with an annual interest rate of 2%, compounded annually. Assume no other costs, and that the specified bank loan and forward contract are the only possible actions.

(a) Identify *all* arbitrage policies with terms of 1 year.

(b) Identify the arbitrage policy with the maximum profit.

Solution:

(a) Let act by taking out a loan only for the money that we require to purchase thingamabobs. In this case, purchasing x thingamabobs today by taking out a loan of $50x$, and then selling the x thingamabobs in 1 year's time results in the profit formula:

$$P(x) = \underbrace{K(x)}_{\text{Sell thingamabobs}} - \underbrace{50 \times (1 + 0.02) \times x}_{\text{Loan repayment}} = K(x) - 51x$$

The profit corresponding to this model is given below:

- $P(0) = 0$
- $P(1) = 49$
- $P(2) = 78$
- $P(3) = 90$
- $P(4) = 87.60$
- $P(5) = 73.05$
- $P(6) = 48.29$
- $P(7) = 15.01$
- $P(8) = -25.36$
- $P(9) = -71.58$
- $P(10) = -122.58$

Hence, the arbitrage policies correspond to $x = 1, 2, 3, 4, 5, 6, 7$. No other policies (values of x) correspond to arbitrage. Depending on interpretation we could also count $x = 0$ (do nothing) as an arbitrage. But strictly speaking, aribtrages require non-zero probabiilty of a profit; choosing $x = 0$ ensures zero profit, and hence would not be an arbitrage under this definition.

If we take alternative action by taking out a loan of the full amount \$500 regardless of x, the possible actions are $x = 0, 1, \ldots, 10$. (Buy x thingamabobs now, sell x of them in 1 year and pay back the loan value of $500 * 1.02 = \$510$.) Our net profit at the end of 1 year corresponds to:

$$P(x) = \underbrace{K(x)}_{\text{Sell thingamabobs}} + \underbrace{50(10 - x)}_{\text{Cash leftover}} - \underbrace{510}_{\text{Loan repayment}}.$$

We can compute the profit for each value of x explicitly, rounded to the nearest cent:

- $P(0) = -10$
- $P(1) = 40$
- $P(2) = 70$
- $P(3) = 83$
- $P(4) = 81.60$

- $P(5) = 68.05$
- $P(6) = 44.29$
- $P(7) = 12.01$
- $P(8) = -27.36$
- $P(9) = -72.58$
- $P(10) = -122.58$

Hence, again the arbitrage policies correspond to $x = 1, 2, 3, 4, 5, 6, 7$. No other policies (values of x) correspond to arbitrage.

(b) From the lists above, we see that $x = 3$ corresponds to the maximum profit for either model, with the first and second models having a profit of \$90 and \$83, respectively.

4.) (Simple options) A *call option* is a contract between a buyer and seller that entitles the buyer the right (but not the requirement) to purchase a fixed amount of stock shares at a specified, "strike" price: The buyer agrees to pay the seller a premium today, which gives the buyer the right to force the seller to sell the shares at the strike price anytime between today and the expiry term of the option. (Hence, the sale completes based on today's agreed upon strike price, not based on market price on the day of the sale.) Assume that the buyer exercises the option only at expiry (not before), and only if it is a rational choice (i.e,. only if the market price at expiry is greater than the strike price).

(a) In entering this contract, briefly explain what the buyer hopes will happen to the future stock price (and why), and what the seller hopes will happen (and why).

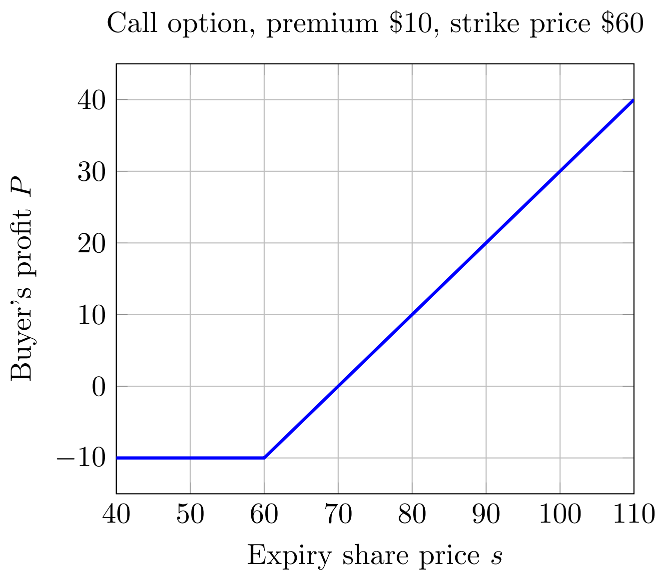
(b) Let ABC be a stock selling on the market today for \$50 per share. Suppose a call option premium is \$10 for a single share of ABC at strike price \$60. Plot the buyer's profit as a function of ABC's expiry share price.

(c) Consider a *put option*, in which the seller pays a premium to gain the right (but not the requirement) to sell stock to the buyer at the strike price. With the same parameters and rational acting as in the previous part, plot the seller's profit as a function of ABC's expiry share price.

<u>Solution:</u>

(a) The buyer hopes that the stock price will rise so that they can buy at the (lower) strike price in the future and immediately sell back to the market for a profit that overcomes the loss in paying the premium. The seller hopes that the stock price will not rise (or fall) so that either the buyer will not exercise the option and the seller can absorb the premium as a profit, or that the stock price rises too little to overcome the profit the seller has gained in the premium.

(b) The buyer's profit $P(s)$ is plotted below as a function of the expiry share price s in dollars.



(c) In this case the seller pays a premium of $10 for forcing a sale at the strike price of $60, i.e., they hope that the stock price falls below $60 so that they can sell to the buyer at $60 and buy back a share from the market at a lower rate, making a profit. The profit $P(s)$ as a function of expiry share price is shown below.