HW #10 due today

Quiz #8 due today (Canvas)

Final exam

- Wed. May 5, 8-10am (Canvas upload window ends @ 11 am)
- same format as midterms (no book, notes, calculator)
  - formula sheet can be used
  - exam & submission on Canvas
- exam is cumulative

Office hours

- Thursday (Apr 29) 4-5pm
- Monday (May 3) 11am-noon

} will update on Canvas calendar.

- Exam on May 5

10.4.3

(a) $u_t = k u_{xx} + c u_x \quad , \quad -\infty < x < \infty$

$\mathcal{F}$ $u(x,0) = f(x)$ (assume $k>0$, c any real number)

$\mathcal{F}$ $U_t = k(-i\omega)^2 U + c(-i\omega) U$, $\quad U = \mathcal{F}\{u\}$, $F = \mathcal{F}\{f\}$

$= (-k\omega^2 - ic\omega) U$

$U(\omega, 0) = F(\omega)$

General sol'n: $U(w,t) = A(w)\exp((-kw^2 - icw)t)$

$$U(w,0) = F(w) \Longrightarrow F(w) = A(w)$$

Solution: $U(w,t) = F(w)\exp(-kw^2 t)\exp(-icwt)$

$$\big\downarrow \mathcal{F}^{-1}$$

$$u(x,t) = ?$$

$$U(w,t) = \underbrace{\left[F(w)\exp(-kw^2 t)\right]}_{G(w,t)} \exp(-icwt)$$

$$g(x,t) = \mathcal{F}^{-1}\{G\}$$

Fourier shift thm.

Formula sheet: $u(x,t) = \mathcal{F}^{-1}\{U\} = g(x-(-ct), t)$

$$= \mathcal{F}^{-1}\{G \cdot \exp(-iwct)\}$$

$$u(x,t) = g(x+ct, t) \quad , \quad g(x,t) = \mathcal{F}^{-1}\{F(w)\exp(-kw^2 t)\}$$

Define $H(w,t) = \exp(-kw^2 t)$

$$\Longrightarrow G(w,t) = F(w) \cdot H(w,t)$$

$$\Longrightarrow g(x,t) = f * h, \quad h(x,t) = \mathcal{F}^{-1}\{H\}.$$

convolution property

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, h(x-s,t)\, ds$$

$$h(x,t) = \mathcal{F}^{-1}\{H\}$$

$$= \mathcal{F}^{-1}\{\exp(-k\omega^2 t)\}.$$

formula sheet

$$= \sqrt{\frac{\pi}{a}}\exp\left(-x^2/4a\right), \quad a = kt$$

$$= \sqrt{\frac{\pi}{kt}}\exp\left(-x^2/4kt\right)$$

$$\Longrightarrow g(x,t) = f * h$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\int_{-\infty}^{\infty} f(s)\exp\left(-\frac{(x-s)^2}{4kt}\right)ds$$

$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(s)\exp\left(-(x-s)^2/4kt\right)ds$$

$$\Longrightarrow u(x,t) = g(x+ct,t)$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(s) \exp\left(-(\cancel{x}x+ct-s)^2/4kt\right) ds}$$

(b) Assume $f(\cancel{t}x) = \delta(x)$

Sketch $u(x,t)$ for "various values" of $t>0$

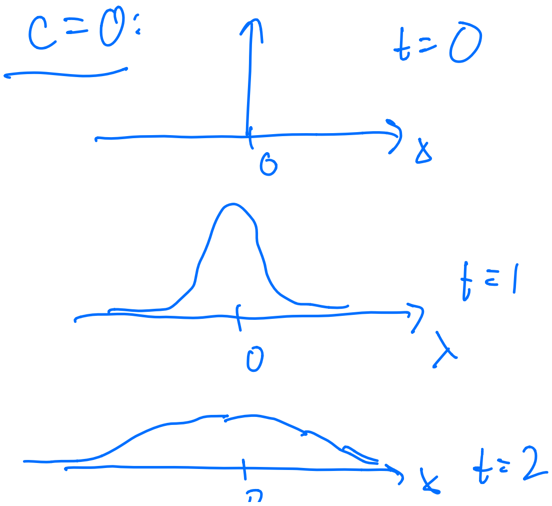
What does the cu_x term in the PDE do?

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \delta(s) \exp\left(-(x+ct-s)^2/4kt\right) ds$$

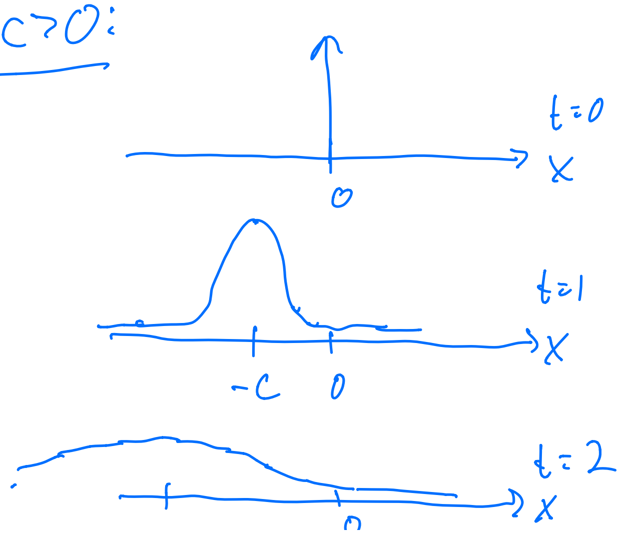
$$= \frac{1}{\sqrt{4\pi k t}} \exp\left(-(x+ct)^2/4kt\right)$$

Note: we've sketched this solution when $c=0$ in class

$c=0$:



$c>0$:



$-2c$

sketches of solution to $u_t = k u_{xx}$
$u(x,0) = \delta(x)$

sketches of solution to "standard" heat equation shifted in space.

Difference is just a $c \cdot t$ shift.

cu_x term shifts the solution by ct units in space.

<u>10.4.12</u>

$$u_t = k u_{xx}, \qquad -\infty < x < \infty$$

$$u(x,0) = f(x). \qquad t > 0$$

$\mathcal{F}$

$$U_t = k(-i\omega)^2 U \quad , \quad U = \mathcal{F}\{u\}$$

$$= k\omega^2 U$$

$\mathcal{F}$

$$U(\omega, 0) = \mathcal{F}\{f\} = F(\omega).$$

General sol'n: $U(\omega,t) = \exp(-k\omega^2 t)\cdot A(\omega)$

$U(\omega,0) = F(\omega) \Longrightarrow A(\omega) = F(\omega)$

solution: $U(\omega,t) = F(\omega)\underbrace{\exp(-k\omega^2 t)}_{H(\omega,t)}$

$= F\cdot H$

convolution prop: $u(x,t) = f*h$, $h = \mathcal{F}^{-1}\{\exp(-k\omega^2 t)\}$

$h(x,t) = \mathcal{F}^{-1}\{\exp(-a\omega^2)\}$ $a = kt$

$= \sqrt{\frac{\pi}{a}}\exp(-x^2/4a)$ (formula sheet)

$= \sqrt{\frac{\pi}{kt}}\exp(-x^2/4kt)$

$u(x,t) = f*h$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, h(x-s,t)\, ds$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}\exp\left(-(x-s)^2/4kt\right)ds$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(s)\exp\left(-(x-s)^2/4kt\right)ds}$$

10.4.7 (a) $u_t = k u_{xxx}$, $-\infty < x < \infty$

$u(x,0) = f(x)$

$\mathcal{F}$: $U_t = k(-i\omega)^3 U = -k\omega^3 i^3 U = ik\omega^3 U$

$U(x,0) = F(\omega)$

$$U(\omega,t) = A(\omega)\exp(ik\omega^3 t)$$

$$\Rightarrow F(\omega) = A(\omega)$$

$$U(\omega,t) = F(\omega)\exp(ik\omega^3 t)$$

$$\text{(i)}\quad u(x,t) = \mathcal{F}^{-1}\{U\} = \int_{-\infty}^{\infty} U(w,t)\, e^{-iwx}\, dw$$

$$\text{(ii)}\quad F(w) = \mathcal{F}\{f\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{u}^{\,f}(x,\cancel{t})\, e^{iwx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{u}^{\,f}(s,\cancel{t})\, e^{iws}\, ds$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} F(w)\, e^{ikw^3 t}\, e^{-iwx}\, dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{iws}\, ds\; e^{ikw^3 t}\, e^{-iwx}\, dw$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)\, e^{iws}\, e^{ikw^3 t}\, e^{-iwx}\, ds\, dw$$

(b) $U(w,t) = F(w)\exp(ikw^3 t)$

Define $G(w,t) = \exp(iktw^3)$

$g(x,t) = \mathcal{F}^{-1}\{G\}$

$\Longrightarrow u(x,t) = f * g$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, g(x-s,\ t)\, ds$$

where $g(x,t) = \int_{-\infty}^{\infty} \exp(iktw^3)\, e^{-jwx}\, dw$