Midterm exam Thurs.

9-11am (on Canvas)

Midterm 2 formula sheet available (otherwise closed book/notes/calculator)

Material: HW#'s 5-7

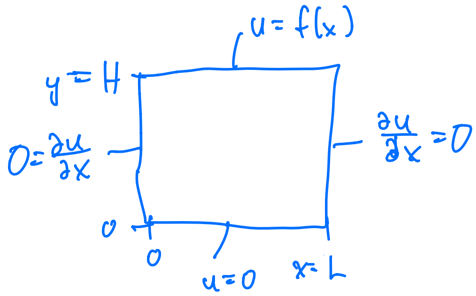
- Laplace's eqn
- Fourier Series
- Wave equation (HW #7 due Today)

No quiz today/tomorrow.

Office hours tomorrow (Wed) @ 10am-11am

2.5.1 (a) $u_{xx}+u_{yy}=0$, $0\le x\le H$, $0\le y\le L$

$\frac{\partial u}{\partial x}(0,y)=0$, $\frac{\partial u}{\partial x}(L,y)=0$, $u(x,0)=0$ $u(x,H)=f(x)$



ansatz: $u(x,\underset{y}{\cancel{t}})=\phi(x)Y(y)$

PDE: $u_{xx}+u_{yy}=0$

$\phi''Y+\phi Y''=0$

$$\frac{\phi''(x)}{\phi(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$$

$\phi''(x)+\lambda\phi(x)=0$

$Y''(y)-\lambda Y(y)=0$

BC's: $\frac{\partial u}{\partial x}(0,y)=0 \longrightarrow \phi'(0)Y(y)=0$

$\longrightarrow \phi'(0)=0$

$\frac{\partial u}{\partial x}(L,y)=0 \longrightarrow \phi'(L)Y(y)=0$

$\longrightarrow \phi'(L)=0$

$u(x,0)=0 \longrightarrow \phi(x)Y(0)=0$

$\longrightarrow Y(0)=0$

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi'(0) = 0$$
$$\phi'(L) = 0$$

$$Y''(y) - \lambda Y(y) = 0, \quad Y(0) = 0$$

Eigenvalue problem: find λ, ϕ that is a nontrivial sol'n to

$$\phi''(x) + \lambda\phi(x) = 0$$
$$\phi'(0) = 0, \quad \phi'(L) = 0$$

char. eqn: $r^2 + \lambda = 0$

$$r = \pm\sqrt{-\lambda}$$

$\underline{\lambda < 0}$: $r = \pm\sqrt{-\lambda}$ real, distinct

$$= \pm\sqrt{|\lambda|}$$

$$\phi(x) = c_1 \exp(\sqrt{|\lambda|}\, x) + c_2 \exp(-\sqrt{|\lambda|}\, x)$$

$$= k_1 \cosh(x\sqrt{|\lambda|}) + k_2 \sinh(x\sqrt{|\lambda|})$$

$$\phi'(x) = k_1 \sqrt{|\lambda|} \sinh(x\sqrt{|\lambda|})$$
$$+ k_2 \sqrt{|\lambda|} \cosh(x\sqrt{|\lambda|})$$

$$\phi'(0) = 0 \Rightarrow k_2 \sqrt{|\lambda|} = 0$$
$$\Rightarrow k_2 = 0$$

$$\phi'(L) = 0 \Rightarrow k_1 \sqrt{|\lambda|} \sinh(L\sqrt{|\lambda|}) = 0$$

$$\downarrow$$
$$\neq 0$$

$$\sinh(L\sqrt{|\lambda|}) \stackrel{?}{=} 0$$

$$e^{L\sqrt{|\lambda|}} - e^{-L\sqrt{|\lambda|}} \stackrel{?}{=} 0$$

since $L\sqrt{|\lambda|} \neq 0$, then

$$\sinh(L\sqrt{|\lambda|}) \neq 0$$

$$\Rightarrow k_1 = 0$$

$$\Rightarrow \phi(x) = 0 \rightarrow \text{no eigenvalues.}$$

$\underline{\lambda = 0}$ $\quad \phi''(x) = 0 \quad , \quad \phi'(0) = 0, \quad \phi'(L) = 0$

$$\phi(x) = c_1 + c_2 x$$

$$\phi'(\overset{x}{\cancel{0}}) = c_2$$

BC's: $\phi'(0) = c_2 = 0$

$\phi'(L) = c_2 = 0$ $\quad \Rightarrow \quad c_2 = 0$

c_1 arbitrary. (choose $c_1 = 1$)

$\lambda_0 = 0, \quad \phi_0(x) = 1$ (eigenpair)

$\underline{\lambda>0}$: $r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$ distinct, imaginary

$$\phi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$$

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(x\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(x\sqrt{\lambda})$$

$$\phi'(0) = 0 \Longrightarrow c_2 = 0$$

$$\phi'(L) = 0 \Longrightarrow c_1\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0$$

$$\sin(L\sqrt{\lambda}) = 0$$

$$L\sqrt{\lambda} = n\pi, \; n = 1, 2 \ldots$$

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2 \ldots$$

at $\lambda = \lambda_n$: $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2 \ldots$

Solution to eigenvalue problem:

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2 \ldots$$

$$\phi_0(x) = 1, \quad \phi_n(x) = \cos\left(\frac{n\pi}{L}x\right) \quad (n > 0)$$

orthogonality: $\int_0^L \phi_n(x)\, \phi_m(x)\, dx$

$$= \begin{cases} 0, & n \neq m \\ L, & n = m = 0 \\ L/2, & n = m > 0 \end{cases}$$

formula sheet

Y ODE: $Y''(y) - \lambda Y(y) = 0$

$$Y(0) = 0$$

at $\lambda = \lambda_n$, $\quad Y_n''(y) - \lambda_n Y_n(y) = 0$

char. eqn: $r^2 - \lambda_n = 0$

$r = \pm\sqrt{\lambda_n}$

if $n=0$: $\lambda_n = 0 \Rightarrow r = \pm 0$ (repeated root)

$Y_0(y) = c_1 + c_2 y$ (also $Y_0'' = 0$)

BC: $Y_0(0) = 0 \Rightarrow c_1 = 0$

$Y_0(y) = c_2 y$

$= a_0 y$ (change c_2 to be called a_0)

if $n>0$: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \Rightarrow r = \pm\left(\frac{n\pi}{L}\right) = \pm\sqrt{\lambda_n}$

$Y_n(y) = c_1 \cosh(y\sqrt{\lambda_n}) + c_2 \sinh(y\sqrt{\lambda_n})$

$Y_n(0) = 0 \rightarrow c_1 = 0$

$Y_n(y) = c_2 \sinh(y\sqrt{\lambda_n})$

$$Y_n(y) = a_n \sinh(y\sqrt{\lambda_n})$$

(rename c_2 to a_n)

$$u(x,y) = \phi(x)\, Y(y)$$

$$u_n(x,y) = \phi_n(x)\, Y_n(y)$$

$$= \begin{cases} \phi_0(x)\, Y_0(y), & n=0 \\ \phi_n(x)\, Y_n(y), & n>0 \end{cases}$$

$$= \begin{cases} a_0 y, & n=0 \\ a_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right), & n>0 \end{cases}$$

Superposition: $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$

$$= a_0 y + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

$$u(x,H) = f(x)$$

$$f(x) = a_0 H + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right)$$

$$f(x) = a_0 H \phi_0(x) + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi H}{L}\right) \phi_n(x)$$

$\downarrow$ multiply by $\phi_m(x)$, integrate

$$\int_0^L f(x)\phi_m(x)\,dx = a_0 H \int_0^L \phi_0(x)\phi_m(x)\,dx$$

$$+ \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi H}{L}\right) \int_0^L \phi_n(x)\phi_m(x)\,dx$$

<u>$m=0$:</u> $\int_0^L f(x)\phi_0(x)\,dx = a_0 H \cdot L$

$$a_0 = \frac{1}{HL}\int_0^L f(x)\,dx$$

<u>$m>0$:</u> $\int_0^L f(x)\phi_m(x)\,dx$

$$= a_m \sinh\left(\frac{m\pi H}{L}\right) \cdot L/2$$

$$a_m = \frac{2}{L \cdot \sinh\left(\frac{m\pi H}{L}\right)} \int_0^L f(x)\phi_m(x)\,dx$$

$$\boxed{\begin{aligned}&\underline{\text{Solution}}: \quad u(x,y) = a_0 y + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)\\ &\text{where} \quad a_0 = \frac{1}{HL}\int_0^L f(x)\,dx\\ &a_n = \frac{2}{L\cdot \sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\end{aligned}}$$

<u>3.2.2</u> (a) $f(x) = x$, $\quad -L \leq x \leq L$

compute the FS coefficients.

$$FS(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

formula sheet:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n>0)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$ (because integrand is odd)

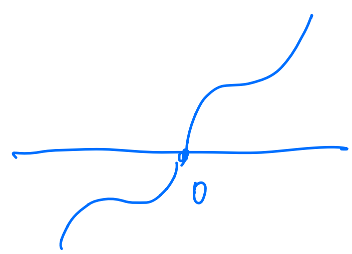
$\left(= \frac{1}{4L} x^2 \Big|_{-L}^{L} \right)$

$a_n = \frac{1}{L}\int_{-L}^{L} x \cos\left(\frac{n\pi x}{L}\right)dx$ (= 0 because the integrand is odd)

IBP: $\int u\,dv = u \cdot v - \int v \cdot du$

$\boxed{f \text{ is odd: } f(x) = -f(-x)}$

(graph is symmetric about origin)



$a_n = \frac{1}{L}\int_{-L}^{L} x \cos\left(\frac{n\pi x}{L}\right)dx$

$u = x \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

$du = dx \qquad dv = \cos\left(\frac{n\pi x}{L}\right)dx$

$$a_n = \frac{L}{n\pi} x \sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{L}{n\pi}\Big[L\cdot \cancel{\sin(n\pi)}^{0} - -L\,\cancel{\sin(-n\pi)}^{0}\Big]$$

$$-\frac{L}{n\pi}\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}$$

$$= \frac{L^2}{(n\pi)^2}\Big[\cos(n\pi) - \underbrace{\cos(-n\pi)}_{=\cos(n\pi)}\Big] \quad (\cos(-x) = \cos x)$$

$$= \frac{L^2}{(n\pi)^2}\Big[\cos(n\pi) - \cos(n\pi)\Big] = 0$$

$a_0 = 0$, $a_n = 0$, $n > 0$,

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

IBP: $u = x$ $\qquad v = \frac{-L}{n\pi}\cos(n\pi x/L)$

$du = dx$ $\qquad dv = \sin\left(\frac{n\pi x}{L}\right)$

$$b_n = -\frac{L}{n\pi} x\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{-L}{n\pi}\left[L\cos(n\pi) \overset{+}{-} -L\overbrace{\cos(-n\pi)}^{//\cos(n\pi)}\right]$$

$$+ \frac{L}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}$$

$$= \frac{-L}{n\pi}2\cdot\underbrace{\cos(n\pi)}_{(-1)^n} + \left(\frac{L}{n\pi}\right)^2\left[\cancelto{0}{\sin(n\pi)} - \cancelto{0}{\sin(-n\pi)}\right]$$

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

$a_n = 0$, $n \geq 0$

$b_n = \frac{2L}{n\pi}(-1)^{n+1}$ $\quad n = 1, 2, 3 \ldots$

Sketch FS: