PDEs on infinite domains

MATH 3150 Lecture 10

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Haberman 5th edition: Section 10.4

The Fourier transform

Given a function $f(x)$ defined on the real line, $-\infty < x < \infty$, the Fourier transform of f is defined as

$$\mathcal{F}\{f\}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad -\infty < \omega < \infty.$$

Given a function $F(\omega)$ defined on the real line, $-\infty < \omega < \infty$, the inverse Fourier transform of F is defined as

$$\mathcal{F}^{-1}\{F\}(x) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad -\infty < x < \infty.$$

We will now use the Fourier transform to solve PDEs on infinite domains.

The heat equation

Using the Fourier transform, compute the solution to the PDE,

$$\begin{aligned} u_t &= k\, u_{xx}, & t > 0,\ \ -\infty < x < \infty \\ u(x,0) &= f(x). \end{aligned}$$

The heat kernel, I

The function,

$$h(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right)$$

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From the previous example, the solution to the heat equation is simply written:

$$u(x,t) = f(x) * h(x,t),$$

where the convolution is taken over the x variable.

The heat kernel, II

Note that the heat kernel is actually a particular solution to the heat equation.

Example

Show that the solution $u(x,t)$ to $u_t = k u_{xx}$ with initial data $u(x,0) = \delta(x)$ is the heat kernel $u(x,t) = h(x,t)$.

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Example

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The heat kernel is an example of a broader class of solutions.

The heat kernel, III

Suppose L is a linear differential operator (in both x and t), and that L is first-order in t.
Let $q(x,t)$ be the solution to the PDE with Dirac mass initial data,

$$\begin{aligned} Lq &= 0, & t > 0,\ \ -\infty < x < \infty \\ q(x,0) &= \delta(x). \end{aligned}$$

Such solutions q are also sometimes called *fundamental solutions* or *impulse responses*.

If $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, then $q(x,t)$ is the heat kernel $h(x,t)$.

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Example

With the notation above, show that the solution u to the PDE

$$\begin{aligned} Lu &= 0, & t > 0, \quad -\infty < x < \infty \\ u(x,0) &= f(x) \end{aligned}$$

is given by $u = f * q$, where the convolution is taken over the x variable.

The wave equation

Using the Fourier transform, compute the solution to the PDE,

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

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Specialize the solution above to the case $g = 0$.

Other PDEs

Using the Fourier transform, compute the solution to the PDE,

$$\begin{aligned} u_t &= c\, u_x, && t > 0, \;\; -\infty < x < \infty \\ u(x,0) &= f(x), \end{aligned}$$