- HW #8 due <u>today</u>
- Quiz due tomorrow (Canvas), available now.
- HW #9 available (due next Tues, Apr. 20).
- Office hours today 1-2pm.

Fourier transform properties

MATH 3150 Lecture 09

April 13, 2021

Haberman 5th edition: Sections 10.3, 10.4

The Fourier transform

Given a function $f(x)$ defined on the real line, $-\infty < x < \infty$, the Fourier transform of f is defined as

$$\mathcal{F}\{f\}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad -\infty < \omega < \infty.$$

$\omega \sim$ eigenvalue λ. (from Fourier Series)

Given a function $F(\omega)$ defined on the real line, $-\infty < \omega < \infty$, the inverse Fourier transform of F is defined as

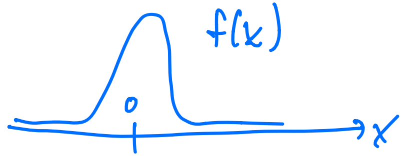
$$\mathcal{F}^{-1}\{F\}(x) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad -\infty < x < \infty.$$

We will spend some time learning about properties of this transform.

ω: "frequency" variable

From HW: $\mathcal{F}$, $\mathcal{F}^{-1}$ are linear operators

Gaussian invariance



A function of the form $f(x) = \exp(-x^2)$ is called a *Gaussian*.

We've seen that

from last lecture

$\beta > 0$

$$\mathcal{F}\left\{\exp\left(-\frac{x^2}{4\beta}\right)\right\} = \sqrt{\frac{\beta}{\pi}}\exp(-\beta\omega^2).$$

Gaussian invariance

A function of the form $f(x) = \exp(-x^2)$ is called a *Gaussian*.

We've seen that

$$\mathcal{F}\left\{\overbrace{\exp\left(-\frac{x^2}{4\beta}\right)}^{f(x)}\right\} = \overbrace{\sqrt{\frac{\beta}{\pi}}\exp(-\beta\omega^2)}^{F(\omega)}.$$

Thus, the Fourier transform of a Gaussian is another Gaussian.

($f \neq F$ in general)

β very large ($\beta >> 1$)

β very small ($0 < \beta << 1$)

Gaussian invariance

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We've seen that

$$\mathcal{F}\left\{\exp\left(-\frac{x^2}{4\beta}\right)\right\} = \sqrt{\frac{\beta}{\pi}}\exp(-\beta\omega^2).$$

Thus, the Fourier transform of a Gaussian is another Gaussian.

Moreover,

- functions that are very concentated in physical space are spread out in frequency space
- functions that are very concentated in frequency space are spread out in physical space

"physical" $\longleftrightarrow$ "frequency"

duality

Shift theorems

From homework #8: Given f with $\mathcal{F}\{f\} = F$,

$$\mathcal{F}\left\{f(x-\beta)\right\} = e^{i\omega\beta} F(\omega),$$

for any real number β.

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Example

Given f, compute $\mathcal{F}\{f(x)e^{i\beta x}\}$ in terms of F, the Fourier transform of f.

$$F = \mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f(x)e^{i\beta x}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\beta x}\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i(\beta+\omega)x}\, dx$$

$$= F(\omega+\beta)$$

Shift theorems

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for any real number β.

Example

Given f, compute $\mathcal{F}\{f(x)e^{i\beta x}\}$ in terms of F, the Fourier transform of f.

I.e.:

- shifts in frequency space correspond to multiplication by complex exponential in physical space
- shifts in physical space correspond to multiplication by complex exponential in frequency space

Differentiation

What does differentiation look like in frequency space?

First easy case: if $f = f(x,t)$ has a Fourier transform with respect to the x variable,

$$\mathcal{F}\{f\} = F(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{f(x,t)}{\cancel{f(x)}} e^{i\omega x}\mathrm{d}x,$$

then

$$\mathcal{F}\left\{\frac{\partial f}{\partial t}\right\} = \frac{\partial F}{\partial t}. = \frac{\partial}{\partial t}\mathcal{F}\{f\}$$

Differentiation

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then

$$\mathcal{F}\left\{\frac{\partial f}{\partial t}\right\} = \frac{\partial F}{\partial t}.$$

The more interesting case: what about differentiation in the x variable?

Example

If $f = f(x)$, compute $\mathcal{F}\{f'\}$ in terms of F. $= \mathcal{F}\{f\}$

$$\mathcal{F}\{f'(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\, e^{i\omega x} dx$$

integrate by parts :

$$u = e^{iwx} \qquad v = f(x)$$

$$du = iwe^{iwx}dx \qquad dv = f'(x)\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\, e^{iwx}dx = \frac{1}{2\pi}\left[e^{iwx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, iwe^{iwx}dx\right]$$

in order for $\int_{-\infty}^{\infty} f(x)\, e^{iwx}dx$ to exist, then

$$\lim_{x\to\pm\infty} |f(x)\, e^{iwx}| = 0.$$

$$\Longrightarrow e^{iwx}f(x)\Big|_{-\infty}^{\infty} = 0.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\, e^{iwx}\, dx = \frac{1}{2\pi}\left[0 - iw\int_{-\infty}^{\infty} f(x)\, e^{iwx}dx\right]$$

$$= -iw\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{iwx}dx$$

$$= -iw\, F(w)$$

$$\mathcal{F}\{f'(x)\} = -iw\,\mathcal{F}\{f\}$$

You can extend this property: if n is a positive integer and $\mathcal{F}\{f^{(n)}(x)\}$ exists, then

$$\mathcal{F}\{f^{(n)}(x)\} = (-iw)^n\,\mathcal{F}\{f\}.$$

Differentiation

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The more interesting case: what about differentiation in the x variable?

Example

If $f = f(x)$, compute $\mathcal{F}\{f'\}$ in terms of F.

Thus,

- differentiation in physical space corresponds to multiplication by ω in frequency space
- differentiation in frequency space corresponds to multiplication by x in physical space

What happens for higher-order derivatives?

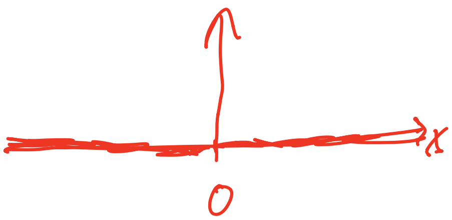
The Dirac delta function

We introduce the Dirac delta function or Dirac mass, $\delta(x)$.

Informally, this is often introduced as

$$\cancel{\delta(x)} \begin{cases} \text{“}\infty\text{”}, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$\delta(x) =$



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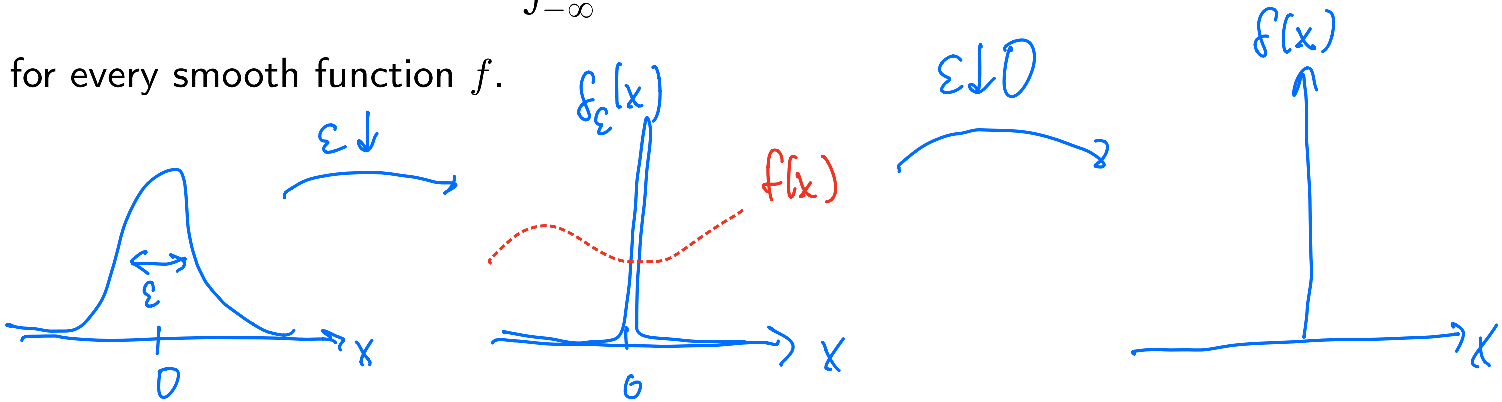
$$\delta(x) = \quad \cancel{\delta(x)} \begin{cases} \text{“}\infty\text{”}, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

More rigorously, this is a “function” satisfying the following property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0),$$

for every smooth function f.



$$\int f(x)\,\delta_\varepsilon(x)\,dx \simeq f(0)$$

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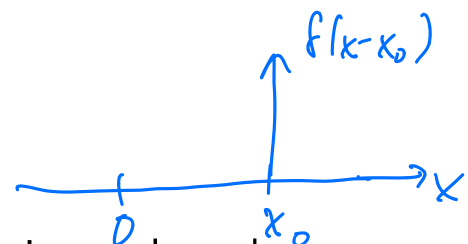

Example

Compute the Fourier transform of $\delta(x - x_0)$, where x_0 is a real number.

$$\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-x_0)\, e^{i\omega x}\, dx$$

$$u = x - x_0 \rightsquigarrow = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(u)\, e^{i\omega(u+x_0)}\, du = \frac{1}{2\pi} e^{i\omega(0+x_0)}$$

$$\Longrightarrow \mathcal{F}\{ f(x-x_0) \} = \frac{1}{2\pi} e^{i w x_0}$$

i.e., $\mathcal{F}\{ f(x) \} = \frac{1}{2\pi}$ (constant)

Apr 15: HW 9 is due Tuesday.

Quiz next Tues/Wed (Canvas)

(Next up: HW #10)

Final exam formula sheet is posted

The Dirac delta function

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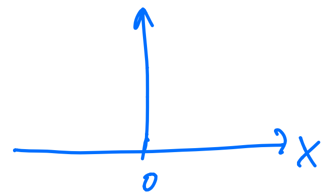
Informally, this is often introduced as

$$f(x) = \cancel{\delta(x)} \begin{cases} \text{“}\infty\text{”}, & x = 0 \\ 0, & x \neq 0 \end{cases}$$



More rigorously, this is a “function” satisfying the following property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0),$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x - x_0)\,dx = f(x_0)$$

for every smooth function f.

Example

Compute the Fourier transform of $\delta(x - x_0)$, where x_0 is a real number.

Thus,

also constants ($e^{i\omega \cdot 0}$)

- Dirac masses in physical space correspond to complex exponentials in frequency space
- Dirac masses in frequency space correspond to complex exponentials in physical space

Convolution, I

The final and perhaps most technical property of Fourier transforms answers the following question:

$F = \mathcal{F}\{f\}$, $G = \mathcal{F}\{g\}$

If (f, F) and (g, G) are Fourier transform pairs, then what is the inverse Fourier transform of $F(\omega)G(\omega)$? What is $\mathcal{F}^{-1}\{F(\omega)G(\omega)\}$?

I.e., what does multiplication in frequency space correspond to in physical space?

Convolution, I

The final and perhaps most technical property of Fourier transforms answers the following question:

If (f, F) and (g, G) are Fourier transform pairs, then what is the inverse Fourier transform of $F(\omega)G(\omega)$?

I.e., what does multiplication in frequency space correspond to in physical space?

Example

Compute $\mathcal{F}^{-1}(FG)$ in terms of f, g.

$$\mathcal{F}^{-1}\{F(\omega)G(\omega)\} = \int_{-\infty}^{\infty} \underbrace{F(\omega)}_{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{i\omega s}\, ds} G(\omega)\, e^{-i\omega x}\, d\omega \qquad (s \neq x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)\, e^{i\omega s}\, ds\;\; e^{-i\omega x}\, G(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s)\, e^{i\omega s}\, e^{-i\omega x}\, G(\omega)\, ds\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s)\, e^{i\omega s}\, e^{-i\omega x}\, G(\omega)\, d\omega\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega(x-s)}\, d\omega\, ds$$

recall: $\mathcal{F}^{-1}\{G\}(x) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega x}\, d\omega$

$$\Rightarrow \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega(x-s)}\, d\omega = g(x-s)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds = \mathcal{F}^{-1}\{F(\omega)\, G(\omega)\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, f(x-s)\, ds$$ (u-substitution) $u = x - s$

Convolution, II

This motivates the following definition:

Definition

Let functions f and g be given. The *convolution* of f and g is the function h defined as

$$h(x) = (f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) f(x - s) \mathrm{d}s$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$$

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Let functions f and g be given. The *convolution* of f and g is the function h defined as

$$h(x) = (f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) f(x-s) \mathrm{d}s$$

Therefore,

- Multiplication in frequency space corresponds to convolution in physical space
- Multiplication in physical space corresponds to convolution in frequency space

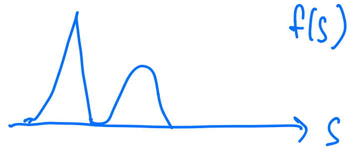
What is convolution?

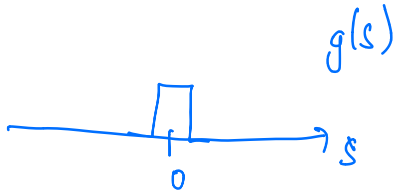
Loosely speaking, convolution is "local" averaging/smoothing.

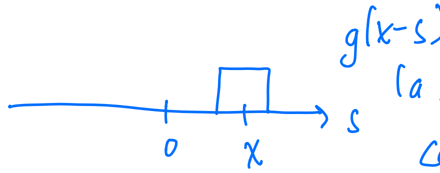
$f*g$: f is "averaged" by g

g is "averaged" by f.

$$f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, g(x-s)$$





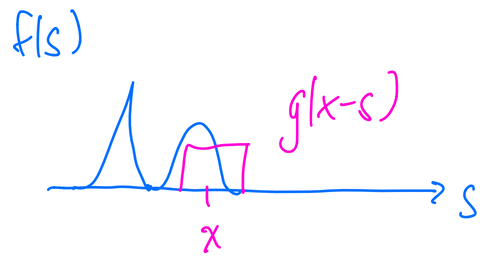


(a square bump centered on x)

what is $\int_{-\infty}^{\infty} f(s) g(x-s)\, ds$?

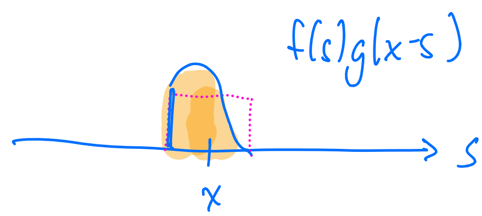
$$\int_{-\infty}^{\infty} f(s) g(x-s)\, ds = 0$$

(no overlap, integrand is 0)



$$\int_{-\infty}^{\infty} f(s) g(x-s)\, ds$$

= area under curve

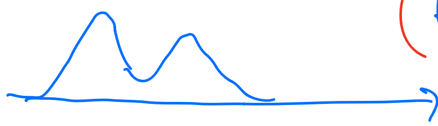


this is a local average of f around x.

$\Longrightarrow$ convolution locally averages f around x.

$\Rightarrow$ convolution smears out graphs.

$(f*g)(x)$



$\cancel{s}\ x$

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Definition

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$$h(x) = (f * g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s) f(x-s)\,\mathrm{d}s$$

Therefore,

- Multiplication in frequency space corresponds to convolution in physical space
- Multiplication in physical space corresponds to convolution in frequency space

Example

Compute the inverse Fourier transform of $F(\omega)e^{i\beta\omega}$ in terms of f using convolutions.

[You have 3 ways to compute this inverse FT: (1) convolution, (2) shift properties, (3) Definition of Inverse FT]

$$F(\omega)e^{i\beta\omega} = F(\omega)G(\omega) \qquad (G(\omega) = e^{i\beta\omega})$$

$$\mathcal{F}^{-1}\{FG\} = f * g$$

$$g = \mathcal{F}^{-1}\{e^{i\beta\omega}\} \quad ?$$

Recall: $\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-x_0) e^{i\omega x} dx$

$$= \frac{1}{2\pi} e^{i\omega x_0}$$

$$\Rightarrow g = \mathcal{F}^{-1}\{e^{i\beta\omega}\} = 2\pi\, \mathcal{F}^{-1}\left\{\frac{1}{2\pi} e^{i\beta\omega}\right\}$$

$$= 2\pi\, \delta(x-\beta)$$

$$\Rightarrow \mathcal{F}^{-1}\{F(\omega)e^{i\beta\omega}\} = f * (2\pi\,\delta(x-\beta))$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, 2\pi\, \delta((x-s)-\beta)\, ds$$

$$= \int_{-\infty}^{\infty} f(s)\, \delta((x-\beta)-s)\, ds$$

$$= \int_{-\infty}^{\infty} f(s)\, \delta(s-(x-\beta))\, ds$$

Dirac mass centered @
$s = x - \beta$.

$= f(x-\beta) = \mathcal{F}^{-1}\{F(\omega)e^{i\beta\omega}\}$

Consider another convolution example:

Given f, F

Let $g(x) = \cos(\nu x)$ ν: positive constant.

$= \frac{1}{2}[e^{i\nu x} + e^{-i\nu x}]$

Using $\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{-i\omega x} d\omega$

$= e^{-i\omega_0 x}$

$\mathcal{F}\{g\} = G(\omega) = \frac{1}{2}[\delta(\omega-\nu) + \delta(\omega+\nu)]$

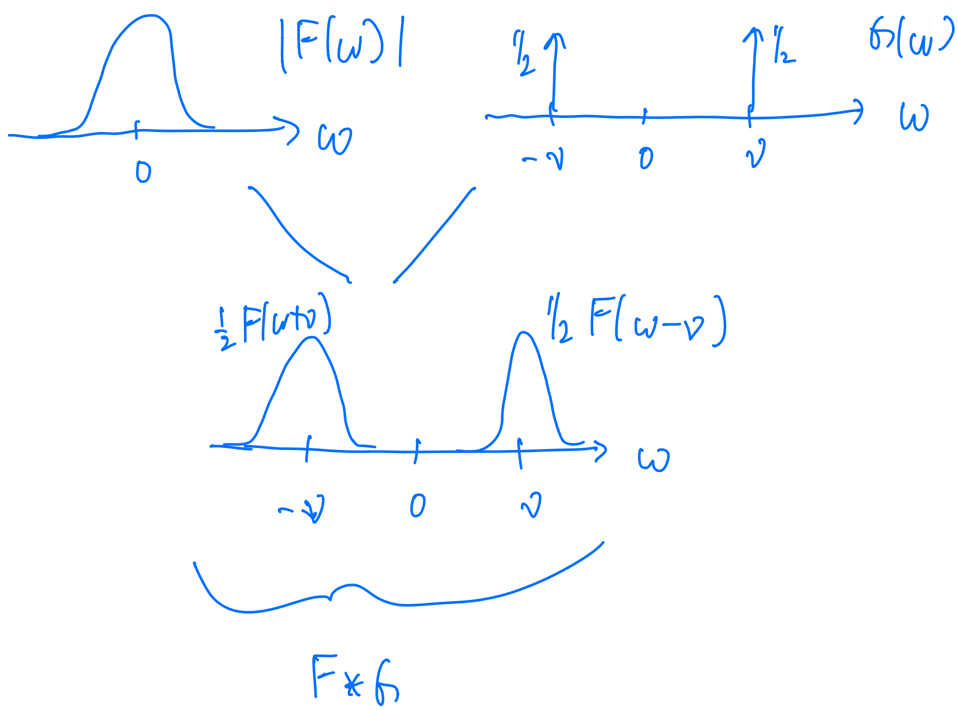
What does $f(x) \cdot g(x)$ correspond to in frequency space?

$f(x)\cos(\nu x)$

formula sheet

$$2\pi \mathcal{F}\{f(x)\cos(\nu x)\} = F * G$$

$$= F * \left(\tfrac{1}{2}\delta(\omega-\nu) + \tfrac{1}{2}\delta(\omega+\nu)\right)$$



$\Longrightarrow$ $f(x)\cos(\nu x)$: takes freq. content of f and shifts it to center on frequency ν.

This is "amplitude modulation" (AM) (radio).

ν : carrier frequency