The Fourier transform and its properties

MATH 3150 Lecture 08

April 6, 2021

Haberman 5th edition: Sections 10.1 - 10.3

PDE's on infinite domains

We have been solving PDEs on bounded spatial domains, e.g.,

$$u_t = u_{xx}, \qquad\qquad -L < x < L.$$

for some finite L.

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for some finite L.

Goal for the rest of the semester: solve PDEs on *unbounded* domains, e.g.,

$$u_t = u_{xx}, \qquad\qquad -\infty < x < \infty.$$

The ideas for bounded domains will extend almost directly to unbounded domains, but the language will look rather different.

(Actually, PDE's are easier on unbounded domains)

The essential change

The main difference on unbounded domains is: we will exchange a Fourier *Series* for a Fourier *Transform*.

In practice, this replaces summations by integration. Given a funciton $f(x)$,

$$\text{Fourier Series} = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$\text{Fourier Transform} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{f(x)}{\cancel{F(\omega)}} e^{i\omega x}\, dx.$$

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$$\text{Fourier Transform} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{F(\omega)}\, e^{i\omega x}\, dx.$$

$f(x)$

- The series is determined by the frequency coefficients a_n, b_n. The transform is determined by the frequency function $F(\omega)$.
- Series: the parameter n is frequency (is discrete). Transform: the parameter ω is frequency (is continuous).
- Series: summation. Transform: integration.

(Riemann Sums)

The next few weeks

Rough outline of next few weeks:

- (1.5 classes) derive relationship between Fourier series and Fourier transform
- (2.5 classes) explore Fourier transform properties
- (2 classes) use Fourier transforms to solve PDEs.

Next: working toward a definition of the Fourier Transform.

Fourier Series $\longrightarrow$ Fourier transform

There are two ways we'll consider to make the connection between a series and a transform.

First method: via a PDE.

$$
\begin{aligned}
u_t &= u_{xx}, & -\infty < x < \infty \\
\lim_{\cancel{x \to \infty}} |u(x,t)| &= 0, & t \geqslant 0
\end{aligned}
$$

$x \to \pm\infty$

"boundary" condition

stability condition

What are the eigenvalues for this problem?

Compare: $\left.\begin{aligned} u(-L,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \xrightarrow{L \uparrow \infty}$ "$u(-\infty, t) = 0$" "$u(\infty, t) = 0$"

Solve using separation of variables:

ansatz: $u(x,t) = \phi(x)T(t)$

$$u_t = u_{xx} \longrightarrow \begin{array}{l} \phi''(x) + \lambda\phi(x) = 0 \\ T'(t) + \lambda T(t) = 0 \\ (\lambda \text{ unknown}) \end{array}$$

$$\text{BC: } \left.|u(\pm\infty, t)| \to 0\right\} \Longrightarrow |\phi(\pm\infty)T(t)| \to 0$$

$$\Downarrow$$

$$\lim_{x\to\pm\infty} |\phi(x)| = 0.$$

<u>ansatz + PDE</u>: $\phi''(x) + \lambda\phi(x) = 0$, $T'(t) + \lambda T(t) = 0$

$$\lim_{x\to\pm\infty} |\phi(x)| = 0$$

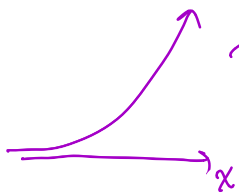
Solve eigenvalue problem: find nontrivial $\phi(x)$ solving

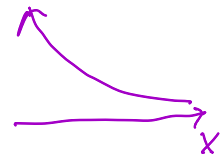
$$\phi''(x) + \lambda\phi(x) = 0$$

$$\lim_{x\to\pm\infty} |\phi(x)| = 0$$

$\underline{\lambda < 0}$: roots of char. eqn are $r = \pm\sqrt{|\lambda|}$ real, distinct

$$\phi(x) = c_1 \exp\left(x\sqrt{|\lambda|}\right) + c_2 \exp\left(-x\sqrt{|\lambda|}\right).$$





$$\lim_{x\to+\infty} |\phi(x)| = 0 \Rightarrow c_1 = 0$$

$$\lim_{x\to-\infty} |\phi(x)| = 0 \Rightarrow c_2 = 0$$

$\Rightarrow$ no $\overset{\phi(x)=0}{\text{eigenvalues}}$ for $\lambda < 0$.

$\underline{\lambda = 0}$: $\phi(x) = c_1 + c_2 x$

$$\lim_{x\to\pm\infty} |\phi(x)| = 0 \rightarrow c_1 = c_2 = 0$$

$\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$

oscillate between $\pm\underline{1}$

In particular, they take on values of 0 many times for large $|x|$.

$\Rightarrow$ these solutions obey something $\underline{\text{like}}$

$$\lim_{x\to\pm\infty} |\phi(x)| = 0$$

(rather they obey $\lim_{x\to\pm\infty} |\phi(x)| < \infty$)

One can show: <u>any</u> $\lambda > 0$ can satisfy BC's.

(with eigenfunction sol'ns $\phi(x) \sim \cos(x\sqrt{\lambda})$
$\sin(x\sqrt{\lambda})$)

so here: eigenvalues λ are <u>not</u> discrete values.

(recall on bounded domains: $\lambda_n \sim \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2 \ldots$)

on unbounded domains: λ is <u>a continuum</u>: all positive real values.

Later we'll see: λ is actually a frequency

Bounded domains: λ (frequency) is discrete

Unbounded domains: λ (frequency) is a continuum.

Fourier Series $\longrightarrow$ Fourier transform

Second method: directly from Fourier series on $[-L, L]$

$$FS(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

What happens as $L \uparrow \infty$? (We'll get the Fourier Transform)

(we get this via an eigenfunction computation, $\lambda = \left(\frac{n\pi}{L}\right)^2$)

Recall: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$(n \geq 1)$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

(formula sheet)

$$FS(x) = \underline{a_0 \cos\left(\frac{0\pi x}{L}\right)} + \sum_{n=1}^{\infty} \underline{\underline{\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}}$$

$$\underline{a_0 \cos\left(\frac{0\pi x}{L}\right)} = a_0 e^{-i0x} \quad \left(\text{since } e^0 = 1 = \cos(0)\right)$$

$$\underline{\underline{a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)}}$$

recall: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\Longrightarrow \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= \frac{a_n}{2}\left[e^{in\pi x/L} + e^{-in\pi x/L}\right] + \frac{b_n}{2i}\left[e^{in\pi x/L} - e^{-in\pi x/L}\right]$$

$$= e^{in\pi x/L}\left[\frac{a_n}{2} + \frac{b_n}{2i}\right] + e^{-in\pi x/L}\left[\frac{a_n}{2} - \frac{b_n}{2i}\right]$$

formula sheet

$$\frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{1}{2Li}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{i}\sin\left(\frac{n\pi x}{L}\right)\right] dx$$

note: $\frac{1}{i} = \frac{i}{i^2} = -i$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\underbrace{\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)}_{e^{-in\pi x/L}}\right] dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$$

define $\omega_n = \frac{n\pi}{L}$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\omega_n x}\, dx$$

$$= \frac{\pi}{L}\left[\underbrace{\frac{1}{2\pi}\int_{-L}^{L} f(x)\, e^{-i\omega_n x}\, dx}_{\text{define } c(\omega_n)}\right]$$

$$= \frac{\pi}{L}\, c(\omega_n)$$

In a similar computation:

$$\frac{a_n}{2} - \frac{b_n}{2i} = \frac{\pi}{L}\, c(-\omega_n)$$

Ie.:

$$FS(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{i\omega_n x}\cdot\frac{\pi}{L}\, c(\omega_n) + e^{-i\omega_n x}\cdot\frac{\pi}{L}\, c(-\omega_n)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{\pi}{L}\,\frac{1}{2\pi}\int_{-L}^{L} f(x)\, e^{i\omega_0 x}\,dx$$

$$(\omega_0 = 0)$$

$$= \frac{\pi}{L} c(\omega_0)$$

$$FS(\cancel{c}\,x) = \frac{\pi}{L} c(\omega_0) \cdot e^{i\omega_0 x} + \frac{\pi}{L} \sum_{n=1}^{\infty} c(\omega_n) e^{i\omega_n x}$$

$$+ \frac{\pi}{L} \sum_{n=-\infty}^{-1} c(\omega_n) e^{i\omega_n x}$$

$$\left(-\omega_n = -\frac{n\pi}{L} = \omega_{-n} \right)$$

$$= \frac{\pi}{L} \sum_{n=-\infty}^{\infty} c(\omega_n) e^{i\omega_n x}$$

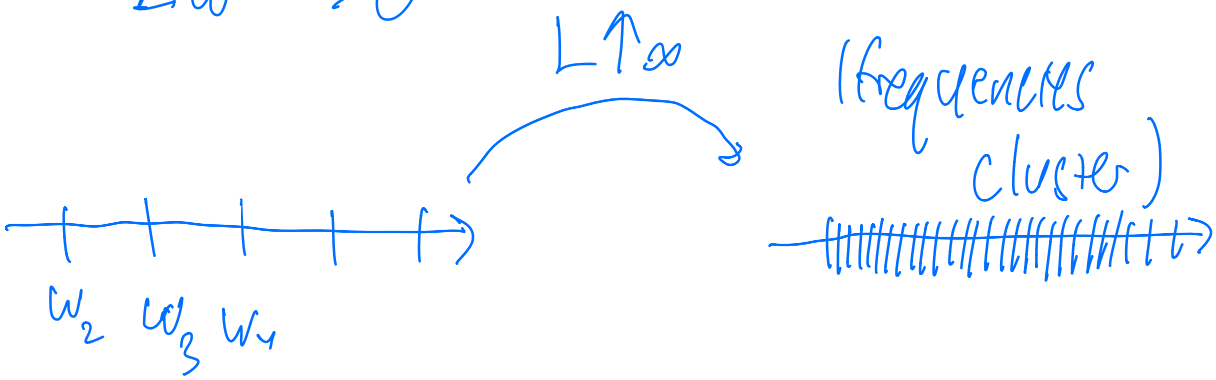
$$c(\omega_n) = \frac{1}{2\pi} \int_{-L}^{L} f(x)\, e^{-i\omega_n x} dx$$

$$\omega_n = \frac{n\pi}{L} \Rightarrow \Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$$

$$FS(x) = \sum_{n=-\infty}^{\infty} c(\omega_n) e^{i\omega_n x} \Delta\omega$$

What happens when $L \uparrow \infty$?

$$\Delta\omega \to 0$$



FS becomes a Riemann sum:

$$FS(x) = \sum_{n=-\infty}^{\infty} c(\omega_n)\, e^{i\omega_n x}\, \Delta\omega$$

$$\Big\downarrow \; L \uparrow \infty, \text{ Riemann sum}$$

$$\int_{-\infty}^{\infty} c(\omega)\, e^{i\omega x}\, d\omega$$

where $c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

This is the Fourier transform.

April 8, 2021

- HW #8 due Tuesday
- Quiz coming Tuesday + Wednesday.
- Office hours this week moved to Friday 12-1pm (April 9)
- Next week: Office hours at Tues at 1-2pm (April 13)

(No office hours Mon. April 12)

The Fourier transform, I

Either method we have discussed results in the following definition:

Definition $F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{iwx}\,dx$

Given a function $f(x)$, the *Fourier transform* of f is $F(\omega)$, defined as

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad -\infty < \omega < \infty$$

(compare against $c(w)$ from previous slides)

Given a function $F(\omega)$, the *inverse Fourier transform* of F is $f(x)$, defined as

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\cancel{\mathrm{d}x}\,dw, \qquad -\infty < x < \infty.$$

(compare against $FS(x)$ from previous slides)

$\mathcal{F}$: operator, "takes the Fourier transform".

$\mathcal{F}^{-1}$: operator, "takes the inverse Fourier transform"

Unlike Fourier Series, $F(w) = \mathcal{F}\{f\}$ is a function on real line

$$f(x) \xrightarrow[\text{coefficients}]{\text{Fourier Series}} a_n,\ b_n \xrightarrow[\sum a_n \cos\left(\frac{n\pi x}{L}\right)+\dots]{\text{series summation}} FS(x) \neq f$$

$-L \leq x \leq L$ $\qquad$ $n = 0, 1, 2 \dots$

in general

$$f(x) \xrightarrow[\mathcal{F}]{\text{Fourier Transform}} F(\omega) \xrightarrow[\mathcal{F}^{-1}]{\text{inverse Fourier transform}} \mathcal{F}^{-1}\{F\} \neq f$$

$-\infty < x < \infty$ $\qquad$ $-\infty < \omega < \infty$

in general

Fourier Transform: $\mathcal{F}\{f\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{i\omega s}\, ds$$

Inverse Fourier transform: $\mathcal{F}^{-1}\{F\} = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$

$$= \int_{-\infty}^{\infty} F(r)\, e^{-irx}\, dr$$

$$\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{irs}\, ds\right) e^{-irx}\, dr \stackrel{?}{=} f(x)$$

(as opposed to $\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx\ e^{-i\omega x}\, d\omega$)

The Fourier transform, I

Either method we have discussed results in the following definition:

Definition

Given a function $f(x)$, the *Fourier transform* of f is $F(\omega)$, defined as

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \cancel{\frac{1}{2\pi} f(x) e^{i\omega x}} \mathrm{d}x, \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \qquad -\infty < \omega < \infty$$

Given a function $F(\omega)$, the *inverse Fourier transform* of F is $f(x)$, defined as

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = \cancel{\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}} \mathrm{d}x, \qquad -\infty < x < \infty.$$

$$= \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

Like in the Fourier series case, it need not be the case that $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$. In general:

$$\frac{1}{2}\left[f(x^{+}) + f(x^{-})\right] = \mathcal{F}^{-1}(\mathcal{F}(f)).$$

In particular: if f is continuous: $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x) = f(x)$.

The Fourier transform, II

if f is continuous

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \qquad = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \cancel{\frac{1}{2\pi} f(x) e^{i\omega x}}\, \mathrm{d}x, \quad f(x) = \mathcal{F}^{-1}\{F\}(x) = \cancel{\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}}\, \mathrm{d}x$$

The Fourier transform is a direct analogue of Fourier series:

The Fourier transform, II

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i w x}\, dx \qquad \text{if } f \text{ is cont.} \qquad = \int_{-\infty}^{\infty} F(w)\, e^{-i w x}\, dw$$

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \cancel{\frac{1}{2\pi} f(x) e^{i\omega x} \mathrm{d}x}, \quad f(x) = \mathcal{F}^{-1}\{F\}(x) = \cancel{\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \mathrm{d}x}$$

The Fourier transform is a direct analogue of Fourier series:

1. Fourier Series: a_n, b_n are the frequency components. Fourier Transform: $F(\omega)$ determines the frequency components

The Fourier transform, II

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \qquad \text{if } f \text{ is cont.} \qquad = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \frac{1}{2\pi} \cancel{f(x) e^{i\omega x}} \mathrm{d}x, \quad f(x) = \mathcal{F}^{-1}\{F\}(x) = \cancel{\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \mathrm{d}x}$$

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1. Fourier Series: a_n, b_n are the frequency components. Fourier Transform: $F(\omega)$ determines the frequency components
2. Fourier Series: The series is formed by summing components over all frequencies. Fourier Transform: The inverse transform is formed by integrating components over all frequences.

The Fourier transform, II

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx \qquad \text{if } f \text{ is cont.} \qquad = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x} d\omega$$

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \cancel{\frac{1}{2\pi} f(x) e^{i\omega x}} \mathrm{d}x, \quad f(x) = \mathcal{F}^{-1}\{F\}(x) = \cancel{\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}} \mathrm{d}x$$

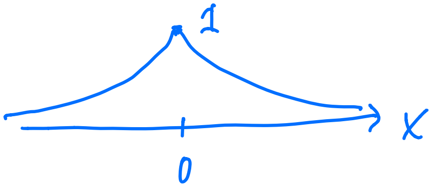
The Fourier transform is a direct analogue of Fourier series:

1. Fourier Series: a_n, b_n are the frequency components. Fourier Transform: $F(\omega)$ determines the frequency components
2. Fourier Series: The series is formed by summing components over all frequencies. Fourier Transform: The inverse transform is formed by integrating components over all frequences.
3. Fourier series: applies over a bounded domain. Fourier Transform: applies over an infinite domain.

Fourier transform examples

Example

Compute the Fourier transform of $f(x) = \exp(-|x|)$.



$$\mathcal{F}\{f\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-|x|)\, e^{i\omega x}\, dx \qquad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{x} e^{i\omega x}\, dx + \frac{1}{2\pi}\int_{0}^{\infty} e^{-x} e^{i\omega x}\, dx$$

$u = -x$

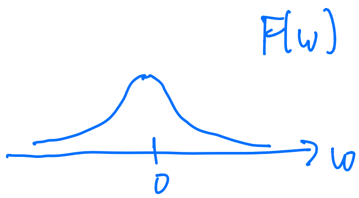
$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{(-1-i\omega)u}\, du + \frac{1}{2\pi}\int_{0}^{\infty} e^{(-1+i\omega)x}\, dx$$

$$= \frac{1}{2\pi}\frac{1}{-1-i\omega} e^{(-1-i\omega)u}\Big|_0^{\infty} + \frac{1}{2\pi}\frac{1}{-1+i\omega} e^{(-1+i\omega)x}\Big|_0^{\infty}$$

$$= \frac{1}{2\pi}\left[\frac{1}{-1-i\omega}(0-1) + \frac{1}{-1+i\omega}(0-1)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+i\omega} + \frac{1}{1-i\omega}\right] = \frac{1}{2\pi}\left[\frac{1-i\omega+1+i\omega}{(1+i\omega)(1-i\omega)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2}{1-(i\omega)^2}\right] = \boxed{\frac{1}{\pi}\frac{1}{1+\omega^2} = F(\omega)}$$



Note: this also allows us to compute $\mathcal{F}^{-1}\{\exp(-|\omega|)\}$.

$$\mathcal{F}^{-1}\{\exp(-|\omega|)\} = \int_{-\infty}^{\infty} e^{-|\omega|} e^{-i\omega x}\, d\omega$$

$$= 2\pi \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|} e^{-i\omega x}\, d\omega$$

$$\overset{\theta=-\omega}{=} 2\pi \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\theta|} e^{i\theta x}\, d\theta$$

$$= 2\pi\, \mathcal{F}\{e^{-|\theta|}\}(x) = 2\pi \cdot \frac{1}{\pi}\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Also notice:

$$\exp(-|x|) \xrightarrow{\ \mathcal{F}\ } \frac{1}{\pi}\,\frac{1}{(1+w^2)}$$

Duality

$$\frac{2}{1+x^2} \xleftarrow{\ \mathcal{F}^{-1}\ } \exp(-|w|)$$

Duality: if $\mathcal{F}\{g\}(w) = G(w)$
then $\mathcal{F}^{-1}\{g\}(x) = G(x)$

up to some multiplicative constants

Fourier transform examples

Example

Compute the Fourier transform of $f(x) = \exp(-|x|)$.

Example

Let $\beta > 0$ be given. Show that the Fourier transform of $f(x) = \exp(-x^2/(4\beta))$ is $F(\omega) = \sqrt{\frac{\beta}{\pi}} \exp(-\beta\omega^2)$.

$$\mathcal{F}\{f\}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/4\beta} e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4\beta} + i\omega x\right) dx$$

exponent: $\frac{-x^2}{4\beta} + i\omega x$ (will complete the square)

$$= \frac{-1}{4\beta}\left[x^2 - 4i\beta\omega x\right]$$

$$= -\frac{1}{4\beta}\left[x^2 - 4i\beta\omega x + (2i\beta\omega)^2 - (2i\beta\omega)^2\right]$$

$$= -\frac{1}{4\beta}\left[(x-2i\beta\omega)^2 - (2i\beta\omega)^2\right]$$

$$= -\frac{1}{4\beta}(x-2i\beta\omega)^2 + \frac{1}{4\beta}\cdot 4\beta^2 i^2 \omega^2$$

$$= -\frac{1}{4\beta}(x-2i\beta\omega)^2 - \beta\omega^2$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{4\beta}(x-2i\beta\omega)^2\right)\exp(-\beta\omega^2)\,dx$$

$$= \frac{\exp(-\beta\omega^2)}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{4\beta}(x-2i\beta\omega)^2\right)dx$$

Define $I(a,b) = \int_{-\infty}^{\infty} \exp(-a(x-b)^2)\,dx$

(So $\cdots = I\left(\frac{1}{4\beta}, 2i\beta\omega\right)$)

(recall: $I(a,b) = \sqrt{\pi/a}$)

$$I(a,b)^2 = \left(\int_{-\infty}^{\infty} \exp(-a(x-b)^2)\,dx\right)\left(\int_{-\infty}^{\infty} \exp(-a(y-b)^2)\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-a(x-b)^2 - a(y-b)^2)\,dx\,dy$$

$x \leftarrow x-b$
$y \leftarrow y-b$

$$\stackrel{}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-a(x^2+y^2))\,dx\,dy$$

use polar coords: $x = r\cos\theta$
$y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{\infty} \exp(-ar^2)\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r\exp(-ar^2)\,dr$$

$u = ar^2$
$du = 2ar\,dr$

$$= 2\pi \frac{1}{2a}\int_0^{\infty} \exp(-u)\,du$$

$$= \pi/a = I^2(a,b) \implies I(a,b) = \sqrt{\pi/a}$$

$$F(w) = \frac{1}{2\pi}\exp(-\beta w^2)\, I\left(\frac{1}{4\beta}, 2i\beta w\right)$$

$$= \frac{1}{2\pi}\exp(-\beta w^2)\cdot\sqrt{\frac{\pi}{1/4\beta}}$$

$$= \frac{1}{\sqrt{\pi}}\sqrt{\beta}\exp(-\beta w^2) = \sqrt{\frac{\beta}{\pi}}\exp(-\beta w^2)$$