Assignments due this week

- HW #6 (Today) ⎫ (Canvas)
- Quiz #5 (Tomorrow) ⎭

Assignment due next week:

- HW #7 (Tuesday)

Midterm #2 next Thursday.

- Review class/session next Tuesday
- New material for midterm #2 ends Thursday.
- Closed book/notes, no calculator
- Formula sheet (#2) can be used during exam
- Format same as midterm #1: exam available/submission window is 9-11am (MT) on Thurs. April 1.
- Material based <u>heavily</u> on homeworks (#5 -7)
  - Laplace's equation
  - Fourier Series
  - Wave equation
- I will <u>not</u> provide a practice exam.
- I will provide solutions to HW #5, 6 on Canvas.

  today early next week.

- I can go over problems from HW #7 say in the review session

Office hours next week (March 29-Apr 2)

- Monday 11am-noon
- Tuesday 9:10 - 10:30 am
- Wednesday 10-11am (special time)
- No office hours on Thursday.

on Canvas calendar.

The wave equation

MATH 3150 Lecture 07

March 23, 2021

Haberman 5th edition: Section 4.1 - 4.4

The wave equation

We've seen two types of PDE's so far:

(heat eqn.) $$u_t = u_{xx}, \qquad u = u(x,t),$$

(Laplace's eqn) $$u_{xx} + u_{yy} = 0, \qquad u = u(x,y).$$

The wave equation

We've seen two types of PDE's so far:

$$u_t = u_{xx}, \qquad u = u(x,t),$$
$$u_{xx} + u_{yy} = 0, \qquad u = u(x,y).$$

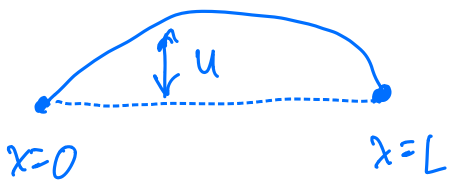
We will consider one more type of PDE in this class, the wave equation,

$$u_{tt} = u_{xx}, \qquad u = u(x,t).$$

Generally: $u_{tt} = c^2 u_{xx}$, c: speed of wave.

Derivation of the wave equation

In one spatial dimension, the wave equation models displacement of an "idealized" string.



Trying to model vertical displacement of the string.

$u(x,t)$: vertical displacement of the string at position x at time t.

We'll assume the string is "idealized"

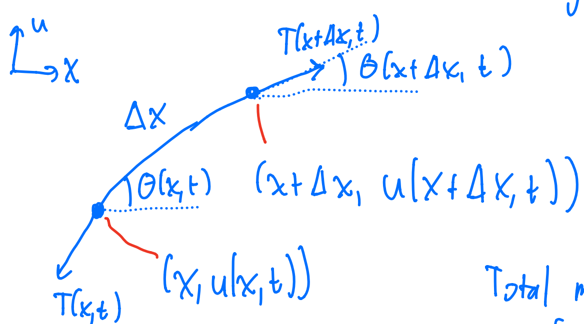
- string has mass.
- ignore gravity
- string is perfectly flexible
- string has ideal tension
- no horizontal displacement of string.

$\rho(x)$: mass density $[{}^{mass}/_{length}]$

$T(\cancel{x})$: internal tensile (restorative) force. [force]

$T(x,t)$ (could also depend on time)

Consider an infinitesimal length of the string:



Ignore gravity

Only forces: tension.

Total mass of this piece of string is $\int_x^{x+\Delta x} \rho(s)\,ds \approx \Delta x\, \rho(x)$

$\Delta x \ll 1$

Displacement of string governed by forces acting on string:

Newton's 2nd law

$$F = ma$$

Consider 2nd law in vertical direction: $a = \dfrac{\partial^2 u}{\partial t^2}$

$$F = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)$$

m : $\Delta x \rho(x)$

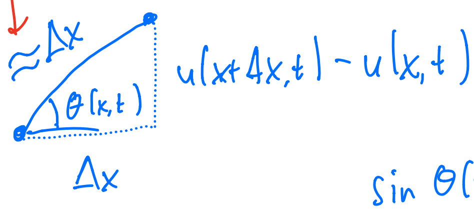
Newton's 2nd law: $F = ma$

$$T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) = \Delta x\, \rho(x)\frac{\partial^2 u}{\partial t^2}$$

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}$$

$$\overset{\Delta x \downarrow 0}{=} \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right]$$

when Δx, u are small.



$$\sin\theta(x,t) = \frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}$$

$$\overset{\Delta x \downarrow 0}{=} \frac{\partial u}{\partial x}$$

Putting these two equations together:

wave equation $\longrightarrow$ $$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\cdot\frac{\partial u}{\partial x}\right)$$

(ρ, T are given string properties)

Assume ρ, T are constants: $\rho(x) = \rho$, $T(x,t) = T$

$$u_{tt} = \frac{T}{\rho}u_{xx} \quad \text{(wave equation)}$$

$c^2 = T/\rho$, c: speed of wave

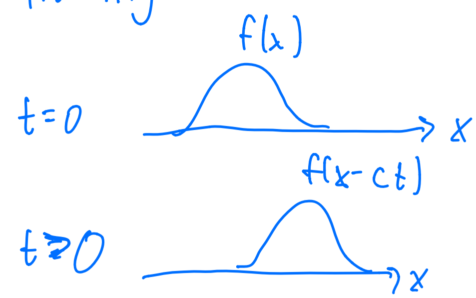
$u_{tt} = c^2 u_{xx}$ (wave equation)

Why is c the wave speed?

$$u_{tt} = c^2 u_{xx}$$

Consider $u(x,t) = f(x-ct)$, f given

traveling wave solution



c is wave speed

Does $f(x-ct)$ solve wave equation?

$$\frac{\partial}{\partial t} f(x-ct) = f'(x-ct) \cdot \frac{\partial (x-ct)}{\partial t}$$

$$= f'(x-ct) \cdot (-c)$$

$$\frac{\partial}{\partial x} f(x-ct) = f'(x-ct) \cdot 1$$

$$(-c)^2 f''(x-ct) = u_{tt} = c^2 u_{xx} = c^2 \cdot f''(x-ct) \quad \checkmark$$

Applications of the wave equation

The wave equation models, unsurprisingly, *wave* phenomena occurring in

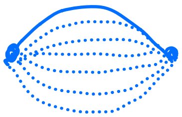
- electromagnetic (light) propagation
- acoustic phenomena
- mechanical stress waves/vibrations/oscillations
- (celestial) gravitational studies
- quantum mechanics

Initial/boundary conditions

Like the heat equation and Laplace's equation, the wave equation requires boundary conditions.

These conditions may be of Dirichlet or Neumann type:

- (Dirichlet) $u(0,t)$, $u(L,t)$: Ends of the string are fixed.



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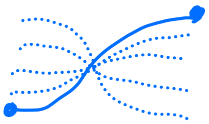

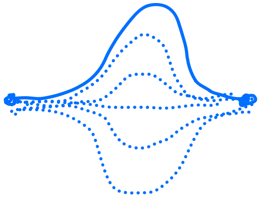

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2 derivatives

Initial conditions: the wave equation is second-order in time. $u_{tt} = u_{xx}$

As a result, we require *two* initial conditions: the value of u and its time derivative:

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Initial displacement

Initial velocity.

Solving the wave equation

The particular wave equation we consider is a linear, homogeneous PDE. Therefore, we can use separation of variables to solve.

Example

Compute the solution $u(x,t)$ to the following PDE:

$$u_{tt} = c^2 u_{xx}, \qquad \text{(c given)}$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{(f, g given)}$$

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

Physically, one can discern *normal modes* and *natural frequencies* from a mathematical solution.

Ansatz: $u(x,t) = \phi(x) T(t)$ (λ unknown)

$$u_{tt} = c^2 u_{xx} \longrightarrow \frac{T''(t)}{c^2 T(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

ODE's: $T''(t) + \lambda c^2 T(t) = 0$

$\phi''(x) + \lambda \phi(x) = 0$

BC's: $u(0,t) = 0 \longrightarrow \phi(0)T(t) = 0$

$\Longrightarrow \phi(0) = 0$

($T(t)=0$ yields trivial sol'n)

$u(L,t) = 0 \longrightarrow \phi(L)T(t) = 0$

$\Longrightarrow \phi(L) = 0$

IC's: no useful information at this point.

Ansatz $\Rightarrow$
$$\phi''(x) + \lambda\phi(x) = 0$$
$$T''(t) + c^2\lambda T(t) = 0$$
$$\phi(0) = 0$$
$$\phi(L) = 0$$

Solve eigenvalue problem: find λ s.t. there is a nontrivial solution ϕ to

$$\phi''(x) + \lambda\phi(x) = 0$$
$$\phi(0) = 0$$
$$\phi(L) = 0$$

solution (need to show work): $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2 \ldots$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) = \sin(x\sqrt{\lambda_n})$$

Orthogonality: $\int_0^L \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} L/2, & n=m \\ 0, & n\neq m \end{cases}$

At $\lambda=\lambda_n$: $T_n''(t)+\lambda_n c^2 T_n(t)=0$

characteristic eqn: $r^2+\lambda_n c^2=0 \quad (T_n(t)=e^{rt})$

$$r=\pm i\sqrt{\lambda_n c^2}$$

$$=\pm i\,\frac{n\pi c}{L},\quad n=1,2,\ldots$$

$$\Rightarrow T_n(t)=a_n\cos(c\sqrt{\lambda_n}\,t)+b_n\sin(c\sqrt{\lambda_n}\,t)$$

$$u_n(x,t)=\phi_n(x)\,T_n(t)=a_n\sin(\sqrt{\lambda_n}\,x)\cos(c\sqrt{\lambda_n}\,t)$$
$$+b_n\sin(\sqrt{\lambda_n}\,x)\sin(c\sqrt{\lambda_n}\,t)$$

How to satisfy $u(x,0)=f$? $\frac{\partial u}{\partial t}(x,0)=g$?

Superposition:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$$

$$=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$
$$+b_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

(General sol'n)

[Cf. heat eqn sol'n $\sim \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right)$]

IC's: $u(x,0)=f(x) \longrightarrow \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

FSS, or orthogonality (multiply by ϕ_m, integrate)

$$\sum_{n=1}^{\infty} a_n \left(\int_0^L \phi_n(x)\,\phi_m(x)\,dx\right) = \int_0^L f(x)\,\phi_m(x)\,dx$$

$$a_m \frac{L}{2} = \int_0^L f(x)\,\phi_m(x)\,dx$$

$$a_m = \frac{2}{L}\int_0^L f(x)\,\phi_m(x)\,dx$$

(compare against FSS coeffs).

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} -a_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) + b_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \overset{?}{=} g(x)$$

multiply by $\phi_m(x)$, integrate.

$$\sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \int_0^L \phi_n(x)\,\phi_m(x)\,dx$$

$$= \int_0^L g(x)\,\phi_m(x)\,dx$$

$$b_m \cdot \frac{m\pi c}{L} \cdot \frac{L}{2} = \int_0^L g(x)\,\phi_m(x)\,dx$$

$$b_m = \frac{2}{m\pi c} \int_0^L g(x)\,\phi_m(x)\,dx$$

Solution: $$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right) \right)$$

where,
$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

To investigate natural frequencies and normal modes, we'll rewrite the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right) \right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sqrt{a_n^2 + b_n^2} \left[\underbrace{\frac{a_n}{\sqrt{a_n^2 + b_n^2}}}_{\cos\theta} \cos\left(\frac{n\pi c t}{L}\right) + \underbrace{\frac{b_n}{\sqrt{a_n^2 + b_n^2}}}_{\sin\theta} \sin\left(\frac{n\pi c t}{L}\right) \right]$$

$$(\cos^2\theta + \sin^2\theta = 1)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b_n}{a_n}$$

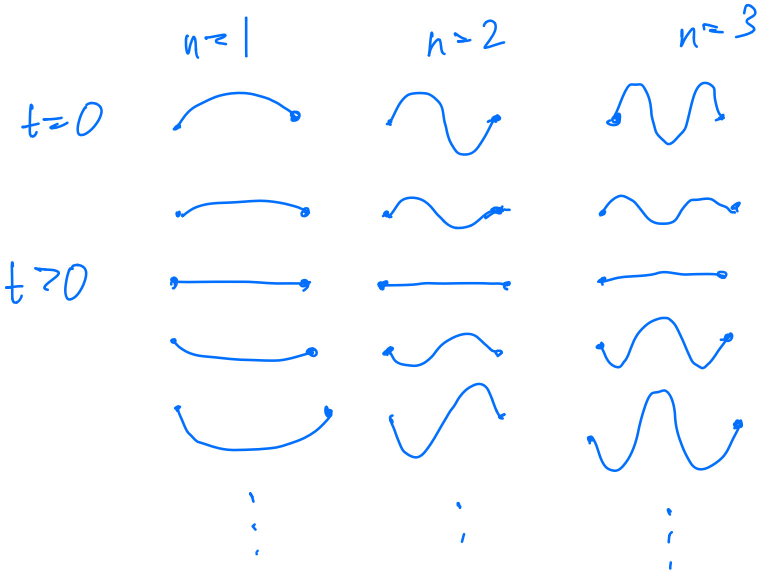
$$\Rightarrow \theta = \arctan(b_n/a_n)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sqrt{a_n^2+b_n^2} \left[\cos\theta \cos\left(\frac{n\pi ct}{L}\right) + \sin\theta \sin\left(\frac{n\pi ct}{L}\right) \right]$$

recall: $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$.

$\left(\alpha = \frac{n\pi ct}{L},\ \beta = \theta\right)$

$$= \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L} - \theta\right), \quad \theta = \arctan\left(b_n/a_n\right)$$



These are modes of vibration.

"Natural frequencies" are temporal frequencies.

Here: $\cos\left(\frac{n\pi ct}{L} - \theta\right)$

$\Downarrow$

Natural frequencies are $\frac{n\pi c}{L}$, $n = 1, 2, \ldots$

Normal modes are individual terms in the summation:

$$\boxed{\text{Normal modes}: \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L} - \theta\right)}$$

Normal modes are also called standing waves