The wave equation

MATH 3150 Lecture 07

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The wave equation

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$$u_{xx} + u_{yy} = 0, \qquad u = u(x,y).$$

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We will consider one more type of PDE in this class, the wave equation,

$$u_{tt} = u_{xx}, \qquad u = u(x,t).$$

Derivation of the wave equation

In one spatial dimension, the wave equation models displacement of an "idealized" string.

Applications of the wave equation

The wave equation models, unsurprisingly, *wave* phenomena occurring in

- electromagnetic (light) propagation
- acoustic phenomena
- mechanical stress waves/vibrations/oscillations
- (celestial) gravitational studies
- quantum mechanics

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These conditions may be of Dirichlet or Neumann type:

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Initial conditions: the wave equation is second-order in time.

As a result, we require *two* initial conditions: the value of u and its time derivative:

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Solving the wave equation

The particular wave equation we consider is a linear, homogeneous PDE. Therefore, we can use separation of variables to solve.

Example

Compute the solution $u(x,t)$ to the following PDE:

$$u_{tt} = c^2 u_{xx},$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

Physically, one can discern *normal modes* and *natural frequencies* from a mathematical solution.