HW #5 due today (Laplace's eqn.)

Quiz #4 due tomorrow (Canvas)

HW #6 due next Tuesday (March 23)

Fourier series

MATH 3150 Lecture 06

March 16, 2021

Haberman 5th edition: Section 3.1 - 3.3

Infinite series

When solving PDEs, we have seen infinite sums of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

In the heat equation: attempt to represent initial data using eigenfunctions.

Recall: $u(x,t) = \sum_{n=0}^{\infty} a_n \exp(-\lambda_n t)\, \phi_n(x)$ $\quad \phi_n$: eigenfunctions

initial data: $u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$

$$= \sum_{n=0}^{\infty} a_n \underbrace{\cos\left(\frac{n\pi x}{L}\right)}$$

eigenfunctions associated to boundary conditions $\phi'(0) = \phi'(L) = 0$

We chose a_n based on orthogonality.

Q: does this actually reproduce f?

Infinite series

When solving PDEs, we have seen infinite sums of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

In the heat equation: attempt to represent initial data using eigenfunctions.

This lecture: can we accurately represent functions using these types of sums?

Fourier series

Let $f(x)$ be a given function for $x \in [-L, L]$.

Definition

The Fourier series of f is defined as,

$$FS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

eigenfunctions associated to periodic BC's.

$\phi(0) = \phi(L)$

$\phi'(0) = \phi'(L)$.

where the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

Where do the formulas for the Fourier coefficients come from?

Orthogonality.

Note: $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \underset{\uparrow}{=} 2\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx$

since cosine is an even function

formula sheet

$$= \begin{cases} 2\cdot L/2, & n=m\neq 0 \\ 2\cdot L, & n=m=0 \\ 2\cdot 0, & n\neq m \end{cases}$$

(this is another orthogonality property.)

Also: $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \underset{\uparrow}{=} 0$

sine is an odd fcn.
cosine is an even fcn.

Fourier coefficients:

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

multiply by $\cos\left(\frac{m\pi x}{L}\right)$, integrate from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2\cdot L\cdot a_m, & \text{if } m=0 \\ L\cdot a_m & \text{if } m>0 \end{cases}$$

$$\Longrightarrow \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \cos\left(\frac{0 \cdot \pi x}{L}\right) dx$$

$$= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

Fourier coeff. formulas.

A similar computation shows:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \begin{cases} L, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Fourier series

Let $f(x)$ be a given function for $x \in [-L, L]$.

Definition

The Fourier series of f is defined as,

$$f(x) \overset{?}{=} FS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

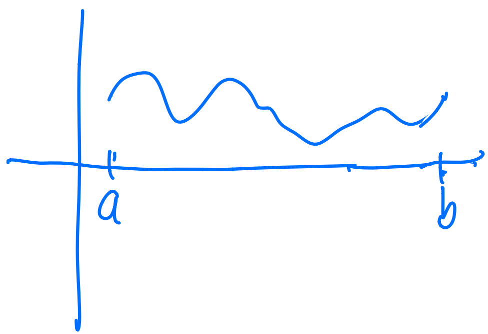
Given f, its Fourier series is, in principle, always computable.
Main question for us: does $FS = f$? Is $FS \approx f$ true?

Smooth functions and discontinuities

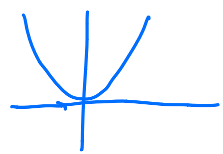
We'll use some terminology to classify types of functions.

- A function f is smooth on the interval $[a, b]$ if f and f' are both continuous for every x in $[a, b]$.

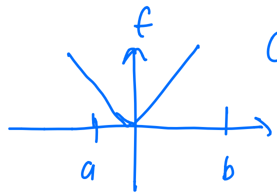
smooth functions :



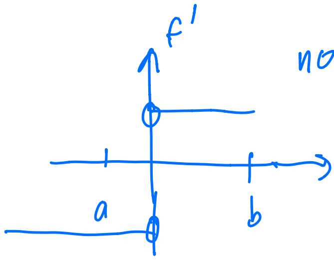
$f(x) = x^2$: smooth for every $[a, b]$



$f(x) = |x|$: not smooth if 0 lies in $[a, b]$



continuous ✓



not continuous X.

Smooth functions and discontinuities

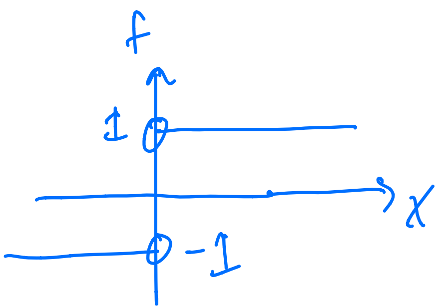
We'll use some terminology to classify types of functions.

- A function f is smooth on the interval $[a,b]$ if f and f' are both continuous for every x in $[a,b]$.
- A function f has jump discontinuity at $x = x_0$ if

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) = f(x_0^+).$$

approach from left

approach from right.



$f(0^-) = -1$

$f(0^+) = +1$

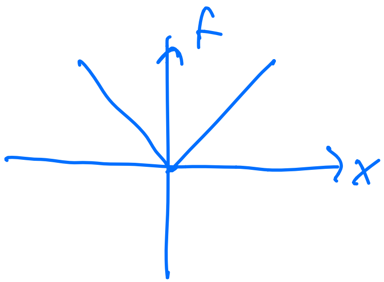
(jump discontinuity)

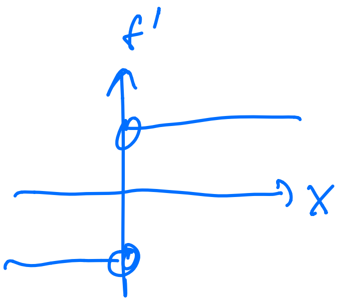
Piecewise smooth functions and periodic extensions

Some more terminology to characterize functions:

- A function f is piecewise smooth on the interval $[a, b]$ if the interval can be broken up into several closed subintervals on each of which f is smooth.

$f(x) = |x|$ is piecewise smooth on $[-1, 1]$.





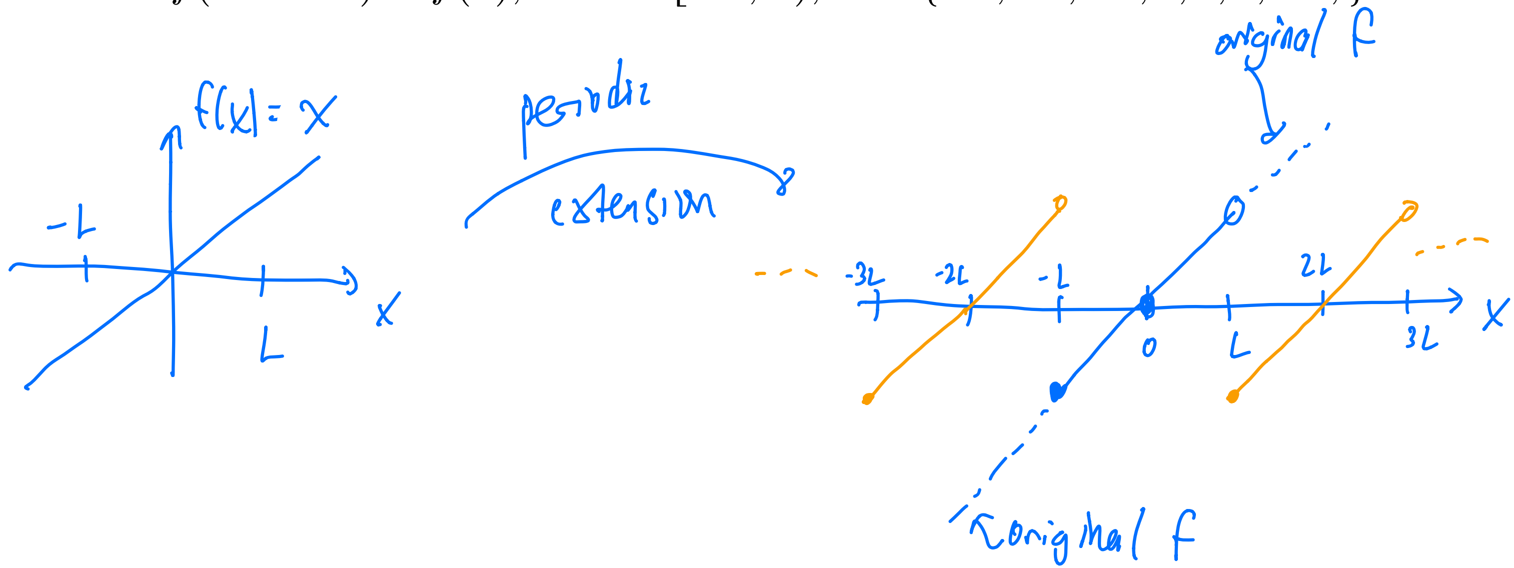
both are smooth on $[-1, 0]$ and $[0, 1]$.

Piecewise smooth functions and periodic extensions

Some more terminology to characterize functions:

- A function f is piecewise smooth on the interval $[a, b]$ if the interval can be broken up into several closed subintervals on each of which f is smooth.
- A function f defined on $[-L, L)$ has a periodic extension on the entire real line defined by

$$f(x + 2nL) = f(x), \qquad x \in [-L, L), \quad n \in \{\ldots, -2, -1, 0, 1, 2, \ldots, \}.$$



Convergence of Fourier series

Given f, the Fourier series is

$$FS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Convergence of Fourier series

Given f, the Fourier series is

$$FS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

We seek to understand the "convergence" of Fourier series. Rigorously, this means that we *truncate* the infinite sums to finite ones,

$$\sum_{n=0}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right),$$

and try to understand what happens as $N \uparrow \infty$.

(Compare limits of sequences, infinite series, power series,
Taylor series ...)

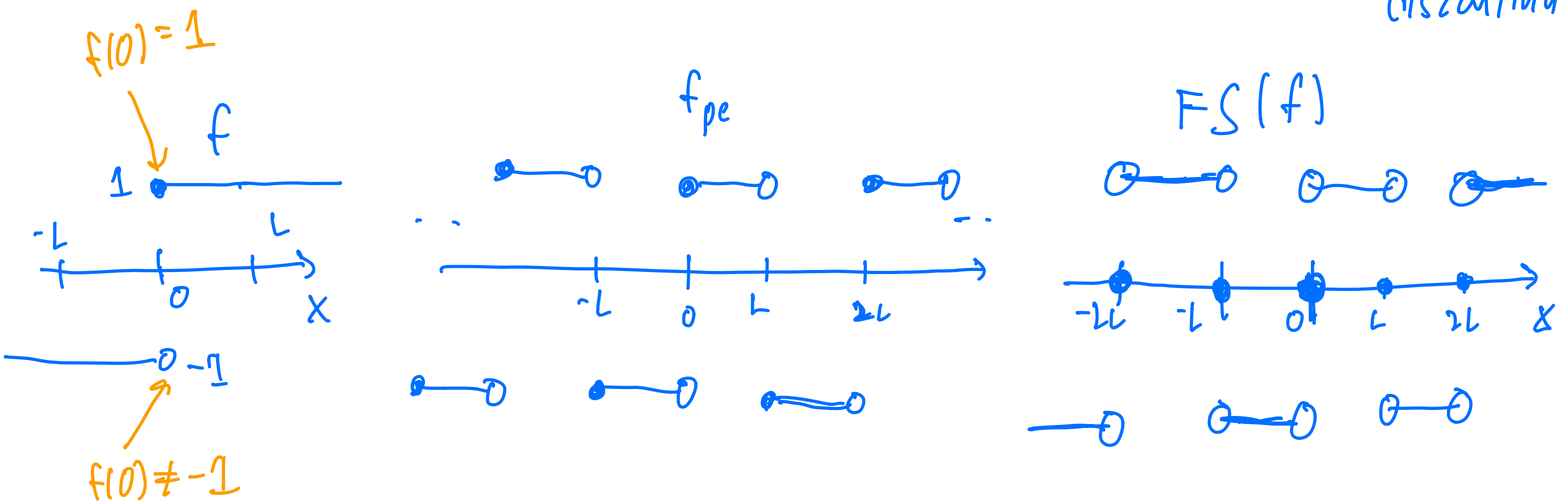
Main result (Convergence of Fourier Series)

Theorem

Let f be a piecewise smooth function on $[-L, L]$; we will also consider its periodic extension, f_{pe}.

Then for every x, the Fourier series of f converges,

- *to f_{pe} at every point x where f_{pe} is continuous*
- *to $\frac{1}{2}\left[f_{pe}(x^+) + f_{pe}(x^-)\right]$ at every point x where f_{pe} is discontinuous.* (with a jump discontinuity)

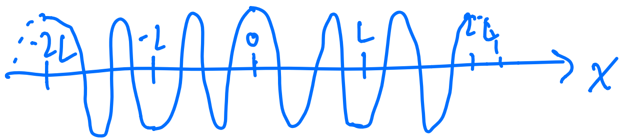


Examples

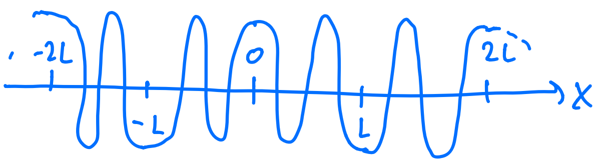
For all the following examples, consider the Fourier series of f over the interval $[-L, L]$, and (a) plot f, (b) plot the periodic extension of f, (c) plot the Fourier series of f, and (d) compute the Fourier coefficients.

Example

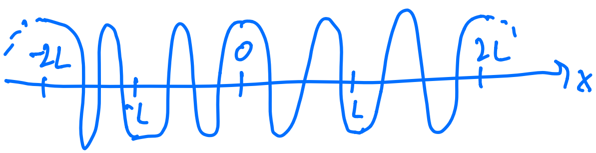
$f(x) = \cos\left(\frac{3\pi x}{L}\right)$

(a) 

f (already periodic)

(b) 

f_{pe}

(c) 

$FS(f)$

(d) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

$= 0$ (because integrand is odd.)

$a_0 = \frac{1}{2L}\int_{\substack{\cancel{0}\\ -L}}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L}\cos\left(\frac{3\pi x}{L}\right)dx = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n \geq 1)$

$= \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{2}{L}\int_{0}^{L}\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$ (integrand is even)

$= \begin{cases} 2/L \cdot L/2, & \text{if } n=3 \\ 2/L \cdot 0, & \text{if } n\neq 3 \end{cases}$ (orthogonality)

$= \begin{cases} 1, & n=3 \\ 0, & n\neq 3 \end{cases}$

$\Rightarrow FS(f) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right) = 1\cdot\cos\left(\frac{3\pi x}{L}\right)$

$= f(x)$

Thursday March 18

- HW #6 is posted (due March 23).
  (Fourier Series)

- Quiz #5 on Tues/Wed next week. (Canvas)

- Midterm #2 is 2 weeks from today.
  - Laplace's equations
  - Fourier Series
  - Wave eqn. (basics, next week)
  - no material directly from midterm #1 topics
  - based on HW (like before)

Recall: Fourier Series on [-L, L]

$$f(x) \cancel{\overset{?}{=}} \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

not in general

|| at points where f_{pe} is continuous

$f_{pe}(x)$

Converges to $\frac{1}{2}\left(f_{pe}(x^+) + f_{pe}(x^-)\right)$ at discontinuities

Examples

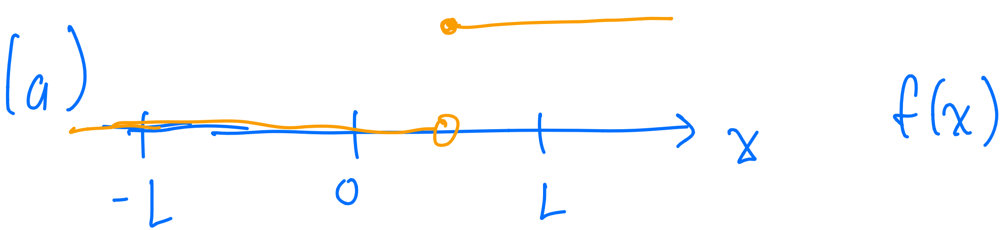
For all the following examples, consider the Fourier series of f over the interval $[-L, L]$, and (a) plot f, (b) plot the periodic extension of f, (c) plot the Fourier series of f, and (d) compute the Fourier coefficients.

Example

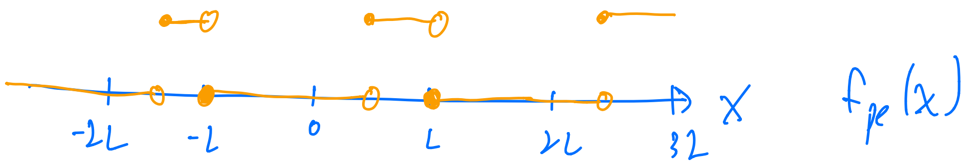
$f(x) = \cos\left(\frac{3\pi x}{L}\right)$

Example

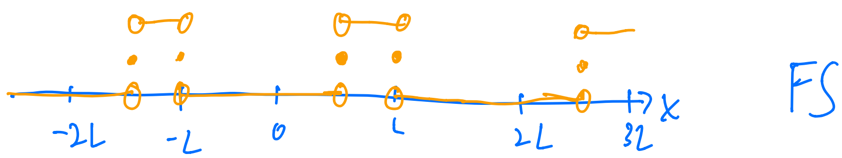
$$f(x) = \begin{cases} 0, & x < L/2 \\ 1, & x \geqslant L/2 \end{cases}$$



(b)



(c)



(d) Compute FS coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{L/2}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\int_{L/2}^{L} 1\,dx = \frac{1}{2L}\cdot\frac{L}{2} = \frac{1}{4}$$

$\underline{n \geq 1:}$ $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{L/2}^{L} \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}\left[\cancel{\sin(n\pi)}^{\,0} - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{-1}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1, & n=1 \\ 0, & n=2 \\ -1, & n=3 \\ 0, & n=4 \\ 1, & n=5 \\ \vdots & \end{cases}$$

$\Longrightarrow$ $a_n = 0$ if n is even.

$$a_{2n-1} = \frac{-1}{(2n-1)\pi}(-1)^{n+1} = \frac{(-1)^n}{(2n-1)\pi}$$

Similar computation for b_n coefficients...

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$, and (a) plot f, (b) plot the periodic extension of f, (c) plot the Fourier series of f, and (d) compute the Fourier coefficients.

Example

$f(x) = \cos\left(\frac{3\pi x}{L}\right)$

Example

$$f(x) = \left\{ \begin{array}{ll} 0, & x < L/2 \\ 1, & x \geqslant L/2 \end{array} \right\}$$

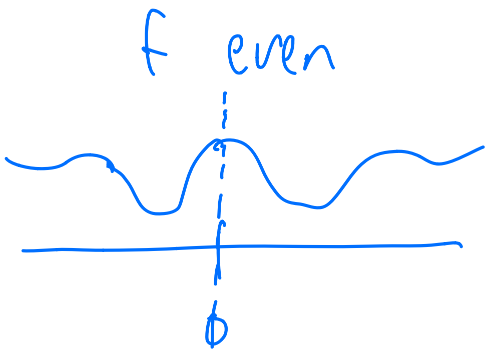
Example

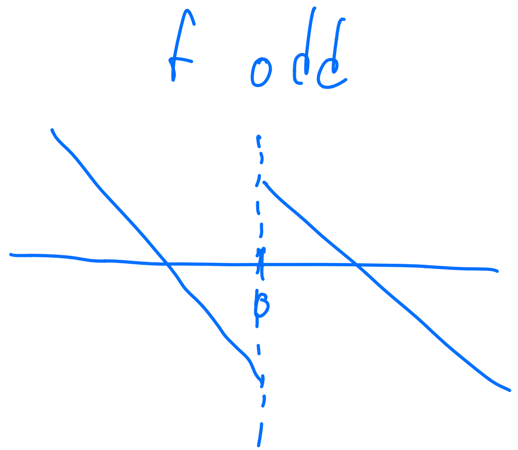
$f(x) = |x|$

Even and odd functions

We will now move a little bit away from our standard Fourier series definitions. Recall:

- A function f is <u>even</u> if $f(x) = f(-x)$ for all real x
- A function f is <u>odd</u> if $f(x) = -f(-x)$ for all real x





Even and odd functions

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Some immediate consequences of this:

- If f is even, its Fourier sine coefficients (b_n) vanish.
- If f is odd, its Fourier cosine coefficients (a_n) vanish.

proof: f is odd: $f(x) = -f(-x)$

$\underline{n \geq 1}$: $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$y = -x$

$dy = -dx$

$$= \frac{1}{L}\left[\int_{0}^{L} \underline{f(-y)}\cos\left(\frac{n\pi y}{L}\right)dy + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

f is odd

$$= \frac{1}{L}\left[\underbrace{-\int_{0}^{L} f(y)\cos\left(\frac{n\pi y}{L}\right)dy} + \underbrace{\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}\right]$$

$= 0.$

Fourier sine series

If f is ~~even~~ odd, then we need only compute the "sine portion" of its Fourier series.

This motivates another definition:

Definition

The Fourier sine series of f on $[0, L]$ is defined as,

FS: Fourier series

$$FSS := \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier sine coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

$FSS \neq FS$

Q: when does the FSS converge to f?

Fourier sine series convergence

Convergence of a Fourier sine series can be understood by directly applying our main theorem about Fourier series.

The result: Let f be a given function.

- Let f_{ope} be the "odd" periodic extension of f: on $[0, L]$
  - $f_{ope}(x) = f(x)$ for x in $[0, L]$.
  - $f_{ope}(x) = -f(-x)$ for x in $[-L, 0]$.
  - f_{ope} is periodically extended outside $[-L, L]$.
- The Fourier sine series of f converges to f_{ope} where f_{ope} is continuous, and to the average of the left- and right-hand limits when it is discontinuous.

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine series on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

$f(x) = x$

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine series on $[0, L]$.

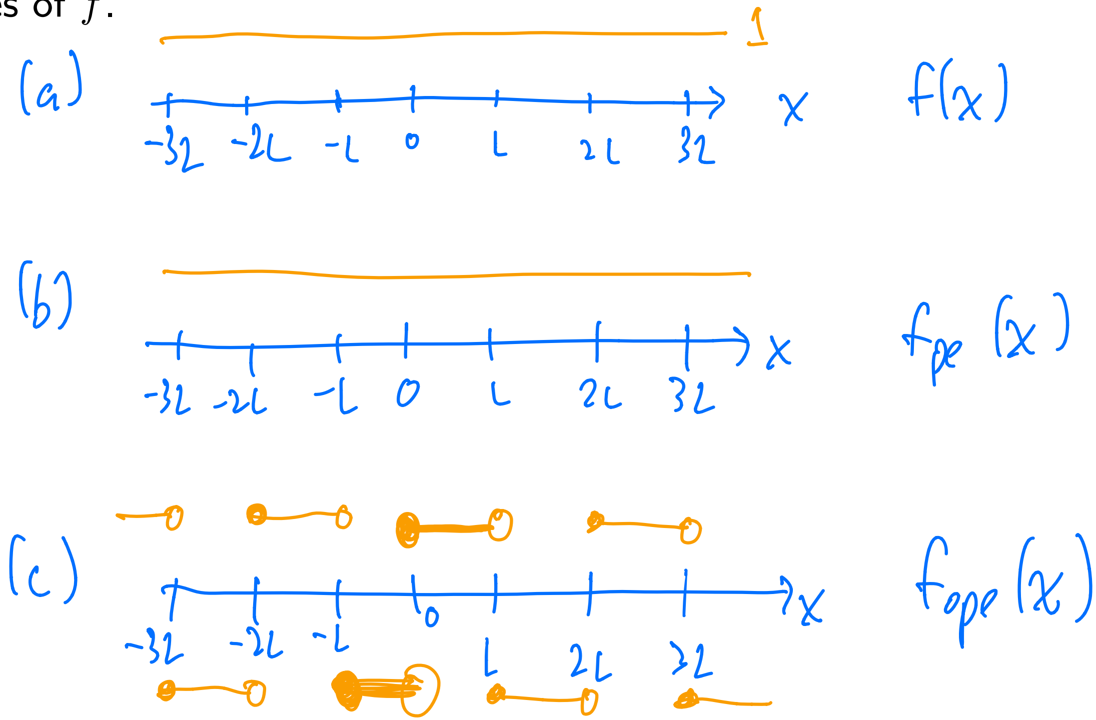
Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

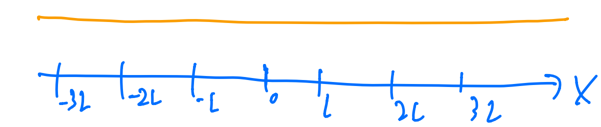
$f(x) = x$

Example

$f(x) = 1$



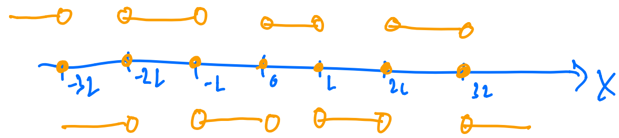
(d)



FS

$\neq$

(e)



FSS

If f is odd: FS = FSS.

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine series on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

$f(x) = x$ $\quad f \neq FS = FSS$

Example

$f(x) = 1$ $\quad f = FS \neq FSS$

Example

$f(x) = \cos(\pi x/L)$ $\quad f = FS$

$FS \neq FSS$

Fourier cosine series

If f is ~~odd~~ even, then we need only compute the "cosine portion" of its Fourier series.

This motivates the definition:

Definition

The Fourier cosine series of f is defined as,

$$\overset{FCS}{F\cancel{S}S} := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

FS ≠ FSS
≠ ≠
FCS (in general)

where the Fourier cosine coefficients are given by

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

Q: when does FCS converge?

Fourier cosine series convergence

Convergence of a Fourier cosine series can be understood by directly applying our main theorem about Fourier series.

The result: Let f be a given function.

- Let f_{epe} be the "even" periodic extension of f:
  - $f_{epe}(x) = f(x)$ for x in $[0, L]$.
  - $f_{epe}(x) = f(-x)$ for x in $[-L, 0]$.
  - f_{epe} is periodically extended outside $[-L, L]$.
- The Fourier cosine series of f converges to f_{epe} where f_{epe} is continuous, and to the average of the left- and right-hand limits when it is discontinuous.

Examples

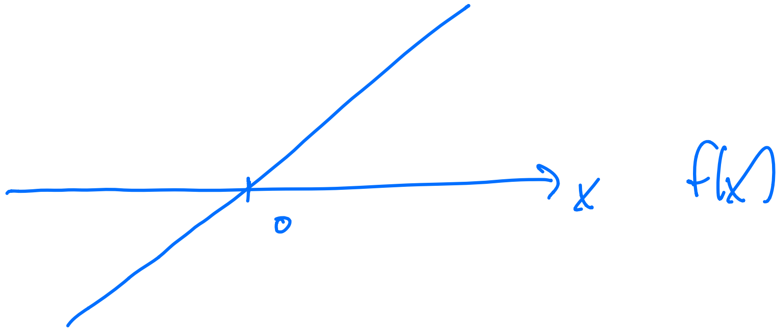
For the following example, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine and cosine series's on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the even periodic extension of f, (e) plot the Fourier series of f, (f) plot the Fourier sine series of f, and (g) plot the Fourier cosine series of f.

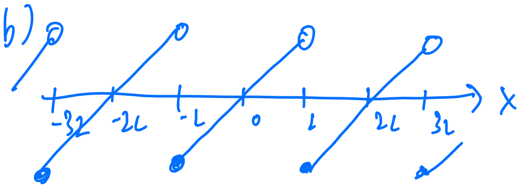
Example

$f(x) = x$

(a)

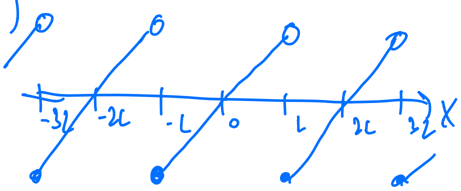


(b)



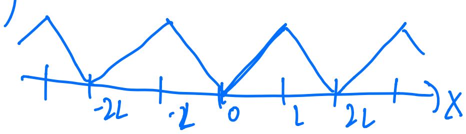
f_{pe}

(c)



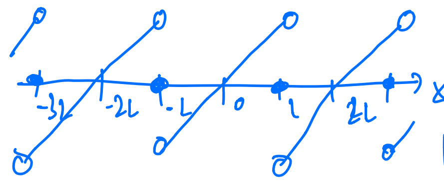
f_{ope}

(d)



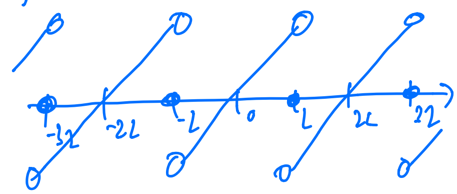
f_{epe}

(e)



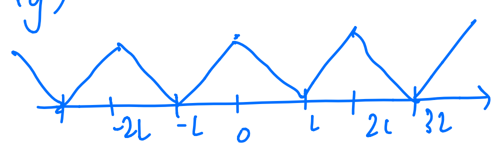
FS

(f)



FSS

(g)



FCS