Fourier series

MATH 3150 Lecture 06

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Haberman 5th edition: Section 3.1 - 3.3

Infinite series

When solving PDEs, we have seen infinite sums of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

In the heat equation: attempt to represent initial data using eigenfunctions.

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In the heat equation: attempt to represent initial data using eigenfunctions.

This lecture: can we accurately represent functions using these types of sums?

Fourier series

Let $f(x)$ be a given function for $x \in [-L, L]$.

Definition

The Fourier series of f is defined as,

$$FS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

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Given f, its Fourier series is, in principle, always computable.
Main question for us: does $FS = f$? Is $FS \approx f$ true?

Smooth functions and discontinuities

We'll use some terminology to classify types of functions.

- A function f is smooth on the interval $[a, b]$ if f and f' are both continuous for every x in $[a, b]$.

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- A function f is smooth on the interval $[a, b]$ if f and f' are both continuous for every x in $[a, b]$.
- A function f has jump discontinuity at $x = x_0$ if

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) = f(x_0^+).$$

Piecewise smooth functions and periodic extensions

Some more terminology to characterize functions:

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Piecewise smooth functions and periodic extensions

Some more terminology to characterize functions:

- A function f is piecewise smooth on the interval $[a, b]$ if the interval can be broken up into several closed subintervals on each of which f is smooth.
- A function f defined on $[-L, L)$ has a periodic extension on the entire real line defined by

$$f(x + 2nL) = f(x), \qquad x \in [-L, L), \;\; n \in \{\ldots, -2, -1, 0, 1, 2, \ldots, \}.$$

Convergence of Fourier series

Given f, the Fourier series is

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We seek to understand the "convergence" of Fourier series. Rigorously, this means that we *truncate* the infinite sums to finite ones,

$$\sum_{n=0}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right),$$

and try to understand what happens as $N \uparrow \infty$.

Main result

Theorem

Let f be a piecewise smooth function on $[-L, L]$; we will also consider its periodic extension, f_{pe}.

Then for every x, the Fourier series of f converges,

- *to f_{pe} at every point x where f_{pe} is continuous*
- *to $\frac{1}{2}\left[f_{pe}(x^+) + f_{pe}(x^-)\right]$ at every point x where f_{pe} is discontinuous.*

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$, and (a) plot f, (b) plot the periodic extension of f, (c) plot the Fourier series of f, and (d) compute the Fourier coefficients.

Example

$f(x) = \cos\left(\frac{3\pi x}{L}\right)$

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$$f(x) = \left\{ \begin{array}{ll} 0, & x < L/2 \\ 1, & x \geqslant L/2 \end{array} \right\}$$

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Example

$f(x) = |x|$

Even and odd functions

We will now move a little bit away from our standard Fourier series definitions. Recall:

- A function f is even if $f(x) = f(-x)$ for all real x
- A function f is odd if $f(x) = -f(-x)$ for all real x

Even and odd functions

We will now move a little bit away from our standard Fourier series definitions. Recall:

- A function f is even if $f(x) = f(-x)$ for all real x
- A function f is odd if $f(x) = -f(-x)$ for all real x

Some immediate consequences of this:

- If f is even, its Fourier sine coefficients (b_n) vanish.
- If f is odd, its Fourier cosine coefficients (a_n) vanish.

Fourier sine series

If f is even, then we need only compute the "sine portion" of its Fourier series.

This motivates another definition:

Definition

The Fourier sine series of f is defined as,

$$FSS := \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier sine coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

Fourier sine series convergence

Convergence of a Fourier sine series can be understood by directly applying our main theorem about Fourier series.

The result: Let f be a given function.

- Let f_{ope} be the "odd" periodic extension of f:
  - $f_{ope}(x) = f(x)$ for x in $[0, L]$.
  - $f_{ope}(x) = -f(-x)$ for x in $[-L, 0]$.
  - f_{ope} is periodically extended outside $[-L, L]$.
- The Fourier sine series of f converges to f_{ope} where f_{ope} is continuous, and to the average of the left- and right-hand limits when it is discontinuous.

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine series on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

$f(x) = x$

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Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

$f(x) = x$

Example

$f(x) = 1$

Examples

For all the following examples, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine series on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the Fourier series of f, and (e) plot the Fourier sine series of f.

Example

$f(x) = x$

Example

$f(x) = 1$

Example

$f(x) = \cos(\pi x/L)$

Fourier cosine series

If f is odd, then we need only compute the "cosine portion" of its Fourier series.

This motivates the definition:

Definition

The Fourier cosine series of f is defined as,

$$FSS := \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where the Fourier cosine coefficients are given by

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

Fourier cosine series convergence

Convergence of a Fourier cosine series can be understood by directly applying our main theorem about Fourier series.

The result: Let f be a given function.

- Let f_{epe} be the "even" periodic extension of f:
  - $f_{epe}(x) = f(x)$ for x in $[0, L]$.
  - $f_{epe}(x) = f(-x)$ for x in $[-L, 0]$.
  - f_{epe} is periodically extended outside $[-L, L]$.
- The Fourier cosine series of f converges to f_{epe} where f_{epe} is continuous, and to the average of the left- and right-hand limits when it is discontinuous.

Examples

For the following example, consider the Fourier series of f over the interval $[-L, L]$ and its Fourier sine and cosine series's on $[0, L]$.

Complete the following (a) plot f, (b) plot the periodic extension of f, (c) plot the odd periodic extension of f, (d) plot the even periodic extension of f, (e) plot the Fourier series of f, (f) plot the Fourier sine series of f, and (g) plot the Fourier cosine series of f.

Example

$f(x) = x$