Announcements:

Class for next 1.5 weeks

- No class next week (Tues. March 7, Thurs. March 9)
- No live class on Thursday this week
  - tomorrow I'll post video to Canvas, and will notify all
  - have new content relevant for next HW assignment

Quizzes

- no quiz today/tomorrow
- no quiz next week
- will be a quiz on March 16-17 (cover topics from <u>this</u> week)

HW assignments

- HW #5 is posted
- due on March 16.

Office hours

- this week: Friday March 5, 12-1pm (special time)
  no OH on Thursday.
- next week: OH as usual, and class time will be OH.

Most of the above: reflected on Canvas calendar.

Laplace's equation

MATH 3150 Lecture 05

March 2, 2021

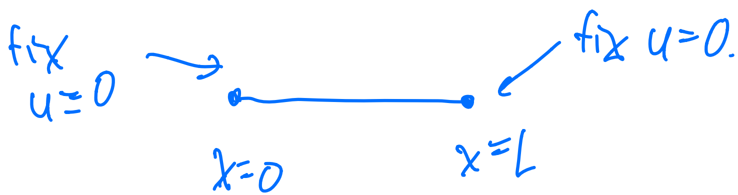
Haberman 5th edition: Section 2.5

The heat equation

The heat equation for $u(x,t)$ is

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$

for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.



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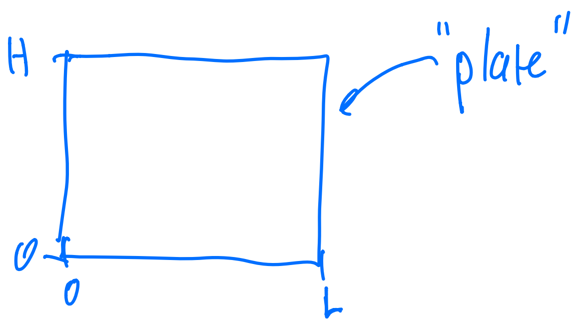
What about for domains that are not (essentially) one-dimensional?

Consider 2D domain (rectangle)



need IC's. $u(x,y,t=0) = f(x,y)$

need BC's: 4 edges

The two-dimensional heat equation

For a two-dimensional rectangle, $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant H$, the heat equation has a similar form,

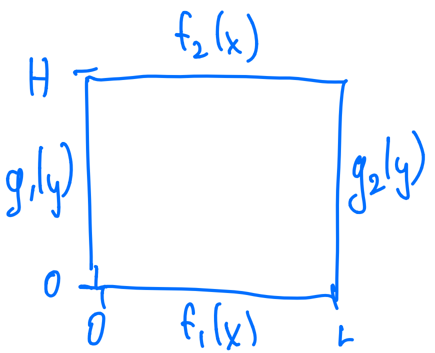
IC.

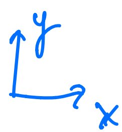
$$u_t = k\, u_{xx} + k\, u_{yy}, \qquad\qquad u(x,y,0) = f(x,y),$$

where $u(x,y,t)$ represents the temperature at location (x,y) at time t.

We also need boundary conditions on the four sides of the rectangle, e.g.,:

$$u(x,0) = f_1(x), \qquad\qquad u(x,H) = f_2(x)$$
$$u(0,y) = g_1(y), \qquad\qquad u(L,y) = \cancel{g_2(x)}\ g_2(y)$$





Neumann conditions: specify "derivative"

$\frac{\partial}{\partial x}$: variability/flux along x-direction

Neumann BC's: partial derivatives in directions normal to boundary.

The two-dimensional heat equation

For a two-dimensional rectangle, $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant H$, the heat equation has a similar form,

$$u_t = k\,u_{xx} + k\,u_{yy}, \qquad\qquad u(x,y,0) = f(x,y),$$

where $u(x,y,t)$ represents the temperature at location (x,y) at time t.

We also need boundary conditions on the four sides of the rectangle, e.g.,:

$$\begin{aligned} u(x,0) &= f_1(x), & \qquad u(x,H) &= f_2(x) \\ u(0,y) &= g_1(y), & \qquad u(L,y) &= g_2(x) \end{aligned}$$

Recall that, in two dimensions, we write $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The operator Δ is the *Laplacian* operator.

In n-dim: $\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$

$u_t = \Delta u = u_{xx} + u_{yy}$.

Heat eqn: $u_t = \Delta u$.

Equilibrium solutions

The equilibrium solution: long-time temperature distribution when transients have settled.

Like in 1D: equilibrium solutions are computed by setting time derivatives to 0.

Equilibrium solutions

The equilibrium solution: long-time temperature distribution when transients have settled.

$$\cancel{u_t}^{\,0} = k\Delta u \rightarrow k\Delta u = 0 \rightarrow \Delta u = 0$$

$$\Delta u = u_{xx} + u_{yy} = 0$$

This is *Laplace's equation*, dictating the steady-state temperature that evolves according to the heat equation.

Equilibrium solutions

The equilibrium solution: long-time temperature distribution when transients have settled.

$$\Delta u = u_{xx} + u_{yy} = 0$$

This is *Laplace's equation*, dictating the steady-state temperature that evolves according to the heat equation.

Unlike the 1D case, solving for equilibrium solutions is much harder in 2D.

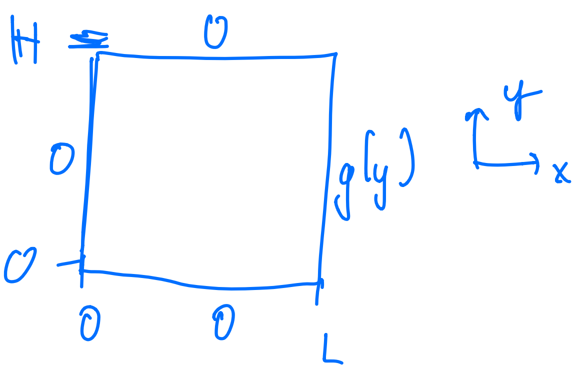
However, it can be tackled using separation of variables, with some minor modifications.

Examples, I

Example

Compute the solution $u(x, y)$ to the following PDE:

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, & 0 < x < L,\ 0 < y < H, \\
u(x,0) &= 0, & u(x,H) = 0 \\
u(0,y) &= 0, & u(L,y) = g(y).
\end{aligned}
$$



Use separation of variables, as we did with the heat equation.

Ansatz: $u(x,y) = \phi(x)\, \Psi(y)$

$u_{xx} = \phi''(x)\, \Psi(y)$

$u_{yy} = \phi(x)\, \Psi''(y)$

PDE $\quad u_{xx} + u_{yy} = 0$

$$\phi''(x)\,Y(y) + \phi(x)\,Y''(y) = 0$$

$$-\phi''(x)\,Y(y) = \phi(x)\,Y''(y)$$

$$-\frac{\phi''(x)}{\phi(x)} = \frac{Y''(y)}{Y(y)} = -\lambda \quad (\lambda \text{ unknown constant})$$

PDE $\longrightarrow$ ODE's: $\quad \phi''(x) - \lambda\phi(x) = 0$

$$Y''(y) + \lambda Y(y) = 0$$

(right) $u(L,y) = g(y) \longrightarrow \phi(L)\,Y(y) = g(y) \longrightarrow$ no condition

$$\left.\begin{array}{lllll} \text{(left)} & u(0,y)=0 & \longrightarrow \phi(0)\,Y(y)=0 & \longrightarrow & \phi(0)=0 \\ \text{(bottom)} & u(x,0)=0 & \longrightarrow \phi(x)\,Y(0)=0 & \longrightarrow & Y(0)=0 \\ \text{(top)} & u(x,H)=0 & \longrightarrow \phi(x)\,Y(H)=0 & \longrightarrow & Y(H)=0 \end{array}\right\} \text{BC's}$$

$$\phi''(x) - \lambda\phi(x) = 0$$

$$Y''(y) + \lambda Y(y) = 0 \quad \bullet$$

$$\phi(0) = 0$$

$$Y(0) = 0 \quad \bullet$$

$$Y(H) = 0 \quad \bullet$$

start with eqn. for which we have 2 boundary conditions

Start with Y ODE to compute eigenvalues:

Compute values of λ and associated functions $Y(y)$ such that there is a nontrivial solution to

$$Y'' + \lambda Y = 0 \qquad , 0 < y < H.$$

$$Y(0) = 0, \quad Y(H) = 0$$

we've solved this eigenvalue problem before, so I'll just write answer:

$$\lambda = \lambda_n = \left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2, 3, \ldots$$

$$Y(y) = Y_n(y) = \sin(y\sqrt{\lambda_n}) = \sin\left(\frac{n\pi y}{H}\right)$$

(In general you need to derive these.)

Orthogonality conditions: $\int_0^H Y_n(y)\, Y_m(y)\, dy = \begin{cases} 0, & n \neq m \\ H/2, & n = m. \end{cases}$

formula sheet

Use $\lambda = \lambda_n$ in other ODE: $\Phi(x)$.

Choose $\lambda = \lambda_n \Rightarrow \Phi$ depends on n, so write as $\Phi_n(x)$.

$$\Phi_n'' - \lambda_n \Phi_n(x) = 0 \qquad (n = 1, 2, \ldots)$$

$$\Phi_n(0) = 0$$

char. eqn : $r^2 - \lambda_n = 0 \rightarrow r = \pm\sqrt{\lambda_n}$

$\lambda_n > 0$, so roots are real, distinct.

$$\phi_n(x) = c_1 \exp(x\sqrt{\lambda_n}) + c_2 \exp(-x\sqrt{\lambda_n})$$

$$\phi_n(0) = 0 \rightarrow c_1 + c_2 = 0 \rightarrow c_2 = -c_1$$

$$\phi_n(x) = c_1 [\exp(x\sqrt{\lambda_n}) - \exp(-x\sqrt{\lambda_n})]$$

recall: $\frac{e^w - e^{-w}}{2} = \sinh(w)$ (hyperbolic sine fcn)

$$= 2c_1 \sinh(x\sqrt{\lambda_n})$$

$$= a_1 \sinh(x\sqrt{\lambda_n}) \quad (a_1 = 2c_1)$$

Then: $u_n(x,y) = \phi_n(x) Y_n(y) = a_1 \sinh(x\sqrt{\lambda_n}) \sin(y\sqrt{\lambda_n})$

$$= a_1 \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$= \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \quad (\text{choose } a_1 = 1)$$

$u_n(x,y)$ solves $\Delta u = 0$, and 3 of 4 BC's

4th BC's: $u(L,y) = g(y)$

To satisfy this: superposition $u(x,y) = \sum_{n=1}^{\infty} b_n u_n(x,y)$ (general sol'n)

for unknown/arbitrary b_n.

$$u(L,y) = \sum_{n=1}^{\infty} b_n u_n(L,y)$$

$$= \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$= g(y)$$

or: $$g(y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi L}{H}\right) \cdot Y_n(y)$$

$\downarrow$ mult. by Y_m, integrate from 0 to H.

$$\int_0^H g(y) Y_m(y)\, dy = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi L}{H}\right) \int_0^H Y_n(y) Y_m(y)\, dy$$

$$= b_m \sinh\left(\frac{m\pi L}{H}\right) \cdot \frac{H}{2} \quad \text{(orthogonality)}$$

$$\Longrightarrow b_m = \underbrace{\frac{2}{H \sinh\left(\frac{m\pi L}{H}\right)} \int_0^H g(y) \sin\left(\frac{m\pi y}{H}\right)}_{\text{all this is known.}} dy$$

Solution: $$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi x}{\cancel{L}\ H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

Examples, II

Example

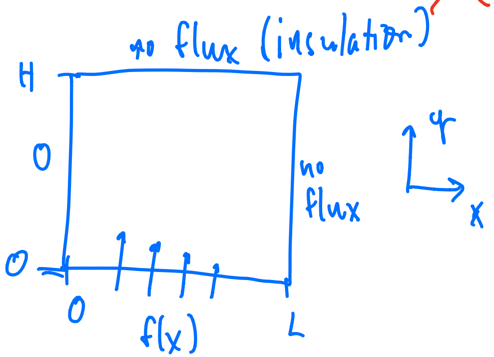
Compute the solution $u(x,y)$ to the following PDE:

$$\Delta u = u_{xx} + u_{yy} = 0, \qquad 0 < x < L,\ 0 < y < H,$$

$$\frac{\partial u}{\partial y}(x,0) = \cancel{0}\ f(x), \qquad \frac{\partial u}{\partial y}(x,H) = 0$$

$$u(0,y) = \cancel{f(x)}\ 0, \qquad \cancel{\frac{\partial u}{\partial y}(L,y) = g(y)}\ 0$$

$$\frac{\partial u}{\partial x}(L,y) = 0$$



Separation of variables

Ansatz: $u(x,y) = \phi(x)Y(y)$

$$u_{xx} + u_{yy} = 0$$

$$\phi'' Y + \phi Y'' = 0$$

$$\frac{\phi''(x)}{\phi(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda \quad \text{(unknown const.)}$$

$$\rightarrow \phi'' + \lambda\phi = 0, \quad Y'' - \lambda Y = 0$$

BC's: $\frac{\partial u}{\partial y}(x, H) = 0 \implies Y'(H)\,\phi(x) = 0$

$\rightarrow Y'(H) = 0$ (don't want $\phi(x) = 0$)

$u(0, y) = 0 \implies \phi(0)\,Y(y) = 0$

$\rightarrow \phi(0) = 0$

$\frac{\partial u}{\partial x}(L, y) = 0 \implies \phi'(L)\,Y(y) = 0$

$\rightarrow \phi'(L) = 0$

$\frac{\partial u}{\partial y}(x, 0) = f(x) \rightarrow \phi(x)\,Y'(0) = f(x)$

$\rightarrow$ no condition.

$\phi''(x) + \lambda\phi(x) = 0 \qquad \phi(0) = 0, \quad \phi'(L) = 0$

$Y''(y) - \lambda Y(y) = 0 \qquad Y'(H) = 0$

Eigenvalue problem: $\phi''(x) + \lambda\phi(x) = 0$

$\phi(0) = 0, \quad \phi'(L) = 0$

char. eqn: $r^2 = -\lambda$, $r = \pm\sqrt{-\lambda}$

$\underline{\lambda<0}$: r values are real, ~~repeated~~ distinct.

$$\phi(x) = c_1 \exp(x\sqrt{-\lambda}) + c_2 \exp(-x\sqrt{-\lambda})$$

recall: $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$= k_1 \cosh(x\sqrt{|\lambda|}) + k_2 \sinh(x\sqrt{|\lambda|})$$

(i.e. $c_1 = \frac{1}{2}k_1 + \frac{1}{2}k_2$, $c_2 = \frac{1}{2}k_1 - \frac{1}{2}k_2$)

$$\phi(0) = 0 \Rightarrow 0 = k_1 \overset{1}{\cancel{\cosh(0)}} + k_2 \overset{0}{\cancel{\sinh(0)}}$$

$$= k_1$$

$$\phi'(x) = k_2\sqrt{|\lambda|}\cosh(x\sqrt{|\lambda|})$$

$$\left(\begin{array}{l} \frac{d}{dx} \cosh x = \sinh x \\ \frac{d}{dx} \sinh x = \cosh x \end{array} \right)$$

$$\phi'(L) = 0 \rightarrow 0 = \underbrace{k_2\sqrt{|\lambda|}}_{\text{strictly positive}}\ \underbrace{\cosh(L\sqrt{|\lambda|})}_{}$$

$\frac{1}{2}(e^{L\sqrt{|\lambda|}} + e^{-L\sqrt{|\lambda|}})$

positive number — positive number

must be strictly positive.

$$\Longrightarrow k_2 = 0$$

$\phi(x) = 0$ is only solution (trivial), so no eigenvalues for $\lambda < 0$.

$\underline{\lambda = 0}$: r values are real, repeated

$r = 0, 0$

$$\phi(x) = c_1 \exp(0) + c_2 x \exp(0) = c_1 + c_2 x$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi'(L) = 0 \Rightarrow c_2 = 0$$

$\longrightarrow \phi(x) = 0$, so $\lambda = 0$ is not an eigenvalue.

$\underline{\lambda > 0}$: $r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

$$\phi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$$

$$\phi(0) = 0 \rightarrow c_1 = 0$$

$$\phi'(L) = 0 \rightarrow \phi'(x) = \sqrt{\lambda}\, c_2 \cos(x\sqrt{\lambda})$$

$$\phi'(L) = 0 \Rightarrow \underbrace{\sqrt{\lambda}}_{\text{not zero}}\, c_2 \cos(L\sqrt{\lambda}) = 0$$

$$\cos(L\sqrt{\lambda}) = 0$$

$$L\sqrt{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2, 3 \ldots$$

$$\phi_n(x) = \sin(x\sqrt{\lambda_n}) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

(take $c_2 = 1$)

formula sheet: $\displaystyle\int_0^L \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \end{cases}$

Back to Y variable: choose $\lambda = \lambda_n$ (for some $n = 1, 2 \ldots$)

$Y = Y_n$

$$Y_n''(y) - \lambda_n Y_n(y) = 0$$

$$Y_n'(H) = 0$$

Char. eqn: $r^2 - \lambda_n = 0$

$r = \pm\sqrt{\lambda_n}$ (real, distinct)

$$Y_n(y) = a_n \cosh(y\sqrt{\lambda_n}) + b_n \sinh(y\sqrt{\lambda_n})$$

$$Y_n'(y) = a_n \sqrt{\lambda_n} \sinh(y\sqrt{\lambda_n}) + b_n \sqrt{\lambda_n} \cosh(y\sqrt{\lambda_n})$$

$$Y_n'(H)=0 \rightarrow a_n \sinh(H\sqrt{\lambda_n}) + b_n \cosh(H\sqrt{\lambda_n}) = 0$$

$$b_n = -a_n \frac{\sinh(H\sqrt{\lambda_n})}{\cosh(H\sqrt{\lambda_n})}$$

$$= -a_n \tanh(H\sqrt{\lambda_n}) \qquad \left(\tanh x = \frac{\sinh x}{\cosh x}\right)$$

$$\Rightarrow Y_n(y) = a_n\left[\overset{\cosh}{\cancel{\sinh}}(y\sqrt{\lambda_n}) - \tanh(H\sqrt{\lambda_n})\,\cancel{\cos}\,\sinh(y\sqrt{\lambda_n})\right]$$

$$u_n(x,y) = \varphi_n(x)\,Y_n(y) = \sin(x\sqrt{\lambda_n})\left[\cosh(y\sqrt{\lambda_n}) - \tanh(H\sqrt{\lambda_n})\cdot \sinh(y\sqrt{\lambda_n})\right]$$

(choose $a_n = 1$)

Superposition: $u(x,y) = \sum_{n=1}^{\infty} d_n\, u_n(x,y)$

$$= \sum_{n=1}^{\infty} d_n \sin(x\sqrt{\lambda_n})\left[\cosh(y\sqrt{\lambda_n}) - \tanh(H\sqrt{\lambda_n})\cdot \sinh(y\sqrt{\lambda_n})\right]$$

general solution.

Last BC: $\frac{\partial u}{\partial y}(x,0) = f(x)$

$$\frac{\partial u}{\partial y}(x,0) = \sum_{n=1}^{\infty} d_n \sin(x\sqrt{\lambda_n})\left[\sqrt{\lambda_n}\sinh(0) - \tanh(H\sqrt{\lambda_n})\sqrt{\lambda_n}\cdot\cosh(0)\right]$$

$$= \sum_{n=1}^{\infty} -\sqrt{\lambda_n}\tanh(H\sqrt{\lambda_n})\, d_n \sin(x\sqrt{\lambda_n})$$

$$\stackrel{?}{=} f(x)$$

multiply by $\varphi_m(x) = \sin(x\sqrt{\lambda_m})$, integrate for $x = 0, \ldots L$.

$$\int_0^L f(x)\,\varphi_m(x)\,dx = \sum_{n=1}^{\infty} -\sqrt{\lambda_n}\tanh(H\sqrt{\lambda_n})\, d_n \cdot \int_0^L \varphi_m(x)\varphi_n(x)\,dx$$

$$= -\sqrt{\lambda_m}\tanh(H\sqrt{\lambda_m})\, d_m \cdot \frac{L}{2}$$

orthogonality

$$\Rightarrow d_m = \frac{-2}{L\sqrt{\lambda_m}\tanh(H\sqrt{\lambda_m})}\int_0^L f(x)\,\phi_m(x)\,dx$$

$$u(x,y) = \sum_{n=1}^{\infty} d_n\, u_n(x,\cancel{t}\,y)$$

$$u_n(x,\cancel{t}\,y) = \phi_n(x)\left[\cosh(y\sqrt{\lambda_n}) - \tanh(H\sqrt{\lambda_n})\cdot \sinh(y\sqrt{\lambda_n})\right]$$

$$\phi_n(x) = \sin(x\sqrt{\lambda_n})$$

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2 \ldots$$

Can we solve Laplace's eqn. with <u>all</u> non-homogeneous BC's.?

E.g.:

$$\Delta u = 0$$

$$u(x,0) = f_1(x) \qquad u(0,y) = f_3(y)$$

$$u(x,H) = f_2(x) \qquad u(L,y) = f_4(y)$$

Standard separation of variables will fail.

To solve this problem: exploit linearity:

write $u(x,y) = \sum_{j=1}^{4} u_j(x,y)$, where $u_j(x,y)$

for fixed j, solves Laplace's equation but

<u>with only 1 non-homogeneous BC</u>.

I.e., $u_1(x,y)$ solves

$$\Delta u_1 = 0$$

$$u_1(x,0) = f_1(x) \qquad u_1(0,y) = 0$$

$$u_1(x,H) = 0 \qquad u_1(L,y) = 0$$

and $u_3(x,y)$ solves

$$\Delta u_3 = 0$$

$$u_3(x,0) = 0 \qquad u_3(0,y) = f_3(y)$$

$$u_3(x,H) = 0 \qquad u_3(L,y) = 0$$

(I.e, u_j has a non-homogeneous BC associated to f_j.)

Note: we know how to solve for u_j individually using separation of vars.

$$\Longrightarrow \quad u(x,y) = \sum_{j=1}^{4} u_j(x,y)$$

(i) solves PDE.

(ii) satisfies all 4 non-homogeneous BC's.