Office hours today: 4-5pm

Hw 2 due Tuesday.

Separation of Variables

MATH 3150 Lecture 04

February 4, 2021

Haberman 5th edition: Sections 2.3-2.4

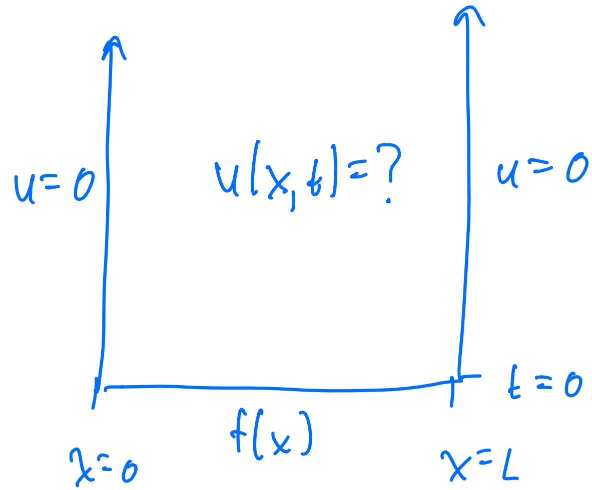
PDEs and the heat equation

Consider the following PDE problem for $u(x,t)$:

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x), \qquad k > 0$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$

for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

Goal: find an (essentially) explicit formula for u.



PDEs and the heat equation

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for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

Our goal for the next 2 weeks is to show how to "solve equations" like the above one.

find a formula for $u(x,t)$

The technique we will use for this is *Separation of Variables*.

(This really can only be used on linear PDE's)

Separation of variables

Separation of variables has three (major) steps:

1. "Separate variables"
   - Use an educated guess to turn PDEs into ODEs
   - Rewrite PDE boundary conditions as ODE conditions
2. Satisfy boundary conditions: compute eigenvalues and eigenfunctions
   - Solve an ODE boundary value problem
   - Compute eigenvalues corresponding to nontrivial (nonzero) solutions
3. Satisfy initial conditions
   - Use superposition to write the general solution to the PDE
   - Compute particular solution satisfying initial data

Step 1: Separate variables

Solve for $u(x, t)$:

$$u_t = k\, u_{xx}, \qquad u(x, 0) = f(x),$$
$$u(0, t) = 0, \qquad u(L, t) = 0,$$

First step: make an $\overbrace{\text{"educated" guess}}^{\text{"ansatz"}}$ for what $u(x,t)$ could look like. (Ignoring initial and boundary conditions)

ansatz: $u(x,t) = T(t)\cdot\phi(x)$, where T and ϕ are unknown.

Near goal: compute T and ϕ.

Again: right now we $\underline{\text{only}}$ consider PDE (not IC, BC).

$u_t = k u_{xx}$.

Note $u(x,t) = 0$ is a solution.

We don't care about it. In particular, $T(t) = 0$ is not helpful (since $u = 0$ then).

If $\varphi(x) = 0$, this is not helpful (then $u = 0$)

New goal: compute "nontrivial" solutions $T(t)$, $\varphi(x)$

not 0.

How to compute T, φ? Need $u(x,t) = T(t)\varphi(x)$ to satisfy PDE: $u_t = k u_{xx}$.

$$u_t = \frac{\partial}{\partial t} u = \frac{\partial}{\partial t}\left(T(t)\varphi(x)\right) = \varphi(x)\frac{\partial}{\partial t}\left(T(t)\right)$$
$$= \varphi(x) T'(t)$$

$$u_{xx} = \frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial x^2}\left(T(t)\varphi(x)\right) = T(t)\frac{\partial^2}{\partial x^2}\left(\varphi(x)\right)$$
$$= T(t)\varphi''(x)$$

$u_t = k u_{xx}$

$\varphi(x) T'(t) = k \varphi''(x) T(t)$

now: separate variables on opposite sides of eqn.

Divide by $u(x,t) = \varphi(x) T(t)$

$$\frac{T'(t)}{T(t)} = k \frac{\varphi''(x)}{\varphi(x)}$$

for convenience.

$$\underbrace{\frac{T'(t)}{kT(t)}}_{\text{depends only on } t} = \underbrace{\frac{\varphi''(x)}{\varphi(x)}}_{\text{depends only on } x.}$$

this equality can only be true if they both equal a constant.

Call this (unknown) constant $-\lambda$.

(Why negative λ? Because λ will be positive.)

$$\frac{T'(t)}{kT(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

$\Longrightarrow$ This results in 2 ODE's:

$$\underline{T'(t) = -\lambda k T(t)} \qquad \underline{\varphi''(x) + \lambda \varphi(x) = 0}$$

To complete this first step: need initial/boundary conditions for these equations.

PDE IC: $u(x,0) = f(x)$

$$\varphi(x)\,T(0) = f(x) \rightarrow T(0) = f(x)/\varphi(x)$$

don't know what $\varphi(x)$ is, this is not a useful condition.

PDE BC: @ $x=0$ $\quad u(0,t) = 0,$

$$\phi(0)\,T(t) = 0.$$

either $T(t) = 0$ or $\phi(0) = 0$

If $T(t) = 0 \longrightarrow$ not interesting since $u = 0$.

Therefore instead, impose $\underline{\phi(0) = 0}$. ✓

PDE BC: @ $x=L$ $\quad u(L,t) = 0$

$$\phi(L)\,T(t) = 0$$

if $T(t) = 0 \Longrightarrow u(x,t) = 0 \rightarrow$ trivial, so not useful.

if $\underline{\phi(L) = 0}$ ✓

To summarize step 1 of separation of variables:

$$\left.\begin{aligned} &u_t = k u_{xx} \\ &u(x,0) = f(x) \\ &u(0,t) = 0 \\ &u(L,t) = 0 \end{aligned}\right\} \xrightarrow{u(x,t) = T(t)\varphi(x)} \left\{\begin{aligned} &T'(t) = -\lambda k T(t) \\ &\varphi''(x) + \lambda\varphi(x) = 0 \\ &\varphi(0) = 0 \\ &\varphi(L) = 0 \end{aligned}\right.$$

Step 2: Boundary conditions (eigenvalues/eigenfunctions)

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$

$T'(t) + \lambda k T(t) = 0$

$\varphi''(x) + \lambda \varphi(x) = 0$

$\varphi(0) = 0$

$\varphi(L) = 0$

Since we have boundary conditions for φ, we'll solve that ODE first.

Find a nontrivial solution φ to the ODE:

$$\begin{cases} \varphi''(x) + \lambda \varphi(x) = 0 \\ \varphi(0) = 0, \quad \varphi(L) = 0 \end{cases} \qquad (\lambda \text{ unknown})$$

→ the equation above has nontrivial solutions only for certain values of λ (that we'll compute).

→ these special values of λ are "eigenvalues". the associated nontrivial solutions φ are "eigenfunctions".

Goal of step 2 of separation of variables: compute eigenvalues and eigenfunctions.

General strategy: exhaust all values of λ.

E.g., take $\lambda = 0$

$$\left.\begin{array}{l} \varphi''(x) = 0 \\ \varphi(0) = 0 \\ \varphi(L) = 0 \end{array}\right\} \quad \varphi(x) = c_1 + c_2 x \quad \text{(integrate twice)}$$

c_1, c_2 are unknown constants of integration.

$\varphi(0)=0 \Longrightarrow \varphi(0)= c_1 + c_2 \cdot 0 = 0 \longrightarrow c_1 = 0$

$\varphi(L)=0 \Longrightarrow \varphi(L)= c_1 + c_2 \cdot L = 0 \longrightarrow 0 + c_2 \cdot L = 0$

$c_2 = 0$

$\Longrightarrow \varphi(x)=0$ is the only solution when $\lambda=0$

$\varphi(x)=0$ is trivial $\Longrightarrow \lambda=0$ is <u>not</u> an eigenvalue.

- Quiz due Wed (midnight)
- Hw (#2) due today (midnight)
- Hw #3 posted by end of day today (due next Tues).
- Midterm exam #1 on Feb. 25.
  - more on Thursday
  - midterm "cheat sheet" posted
  - practice midterm posted.
- Office hours thursday (just this week) moved to Friday ~~2-3pm~~ 1-2pm

ODE problem: $\varphi''(x) + \lambda \varphi(x) = 0$ ($\lambda = 0$ not an eigenvalue)

$\varphi(0) = 0$

$\varphi(L) = 0$

Recall from ODE's: This eqn is a second-order, constant-coefficient, linear, homogeneous ODE.

Method of solution for this type of ODE: solutions via characteristic equation.

Idea: ansatz: $\varphi(x) = \exp(rx)$, r unknown constant.

$$\varphi''(x) = r^2 \exp(rx)$$

$$\varphi''(x) + \lambda \varphi(x) = 0 \longrightarrow (r^2 + \lambda)\exp(rx) = 0$$

$\exp(rx)$ is never zero.

$$\Longrightarrow r^2 + \lambda = 0$$

$r = \pm\sqrt{-\lambda}$ (where $\sqrt{-\lambda}$ can be complex)

Therefore, if $\pm\sqrt{-\lambda}$ are two distinct numbers, then the general solution to $\varphi'' + \lambda\varphi = 0$ is

$$\phi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$$

$$r_1 = +\sqrt{-\lambda}, \quad r_2 = -\sqrt{-\lambda}$$

If $+\sqrt{-\lambda} = -\sqrt{-\lambda}$ (i.e. $\lambda = 0$), then the general solution is

$$\phi(x) = c_1 \exp(r_1 x) + c_2 x \exp(r_1 x)$$

<u>Recall</u>: if r_1, r_2 are real, then

$$\phi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$$

if r_1, r_2 are complex numbers:

they must be complex conjugates (for real-valued ODE's)

$$r_1 = \sigma + i\omega \qquad r_2 = \sigma - i\omega$$

$$i = \sqrt{-1}, \quad \sigma = \mathrm{Re}\{r_1\}$$

$$\omega = \mathrm{Re}\{r_1\}.$$

$$\phi(x) = c_1 \exp((\sigma + i\omega)x) + c_2 \exp((\sigma - i\omega)x)$$

(using $\exp(i\theta) = \cos\theta + i\sin\theta$ (Euler's formula)

$$\phi(x) = k_1 \exp(\sigma x)\cos(\omega x) + k_2 \exp(\sigma x)\sin(\omega x)$$

(where k_1, k_2 are just different constants)
We'll prefer this real-valued solution when r_1, r_2 are complex.

general solution

$\varphi(x) \xrightarrow[r_1, r_2 \text{ real}]{r_1 \neq r_2}$ exponentials

$\varphi(x) \xrightarrow{r_1 = r_2 = 0} c_1 \exp(0x) + x\, c_2 \exp(0x) = c_1 + c_2 x$

$\varphi(x) \xrightarrow{r_1, r_2 \text{ complex}}$ trigonometric.

Strategy: divide analysis into cases of λ when

1.) $r_1 = r_2 = 0 \quad (\lambda = 0)$ ✓

$r_1, r_2 = \pm\sqrt{-\lambda}$ 2.) $r_1 \neq r_2$ are real $(\lambda < 0)$

3.) $r_1 \neq r_2$ are complex $(\lambda > 0)$

$\underline{\lambda = 0}$ (we already did this!)

$$r_{1,2} = \pm\sqrt{-\lambda} = 0, 0$$

$$\varphi(x) = c_1 \underbrace{\exp(0x)}_{1} + c_2 x \exp(0x) = c_1 + c_2 x$$

$$\left.\begin{array}{l} \varphi(0) = 0 \\ \varphi(L) = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0 \quad \text{(from before)}$$

$\varphi(x) = 0 \longrightarrow$ trivial, so $\lambda = 0$ <u>not</u> an eigenvalue.

$\underline{\lambda < 0}$: $r_1, r_2 = \underbrace{\pm\sqrt{-\lambda}}_{\text{real, nonzero.}}$ $\qquad r_1 = \sqrt{-\lambda}$

$r_2 = -\sqrt{-\lambda}$

$$\varphi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$$

$$= c_1 \exp(\sqrt{-\lambda}\, x) + c_2 \exp(-\sqrt{-\lambda}\, x)$$

$\varphi(0) = 0$ & $\varphi(L) = 0$

$$\varphi(0) = 0 \Longrightarrow 0 = c_1 \exp(0) + c_2 \exp(0)$$

$$= c_1 + c_2$$

$$c_2 = -c_1$$

$$\varphi(L) = 0 \Longrightarrow 0 = c_1 \exp(\sqrt{-\lambda}\, L) - c_1 \exp(-\sqrt{-\lambda}\, L)$$

$$c_1 \underbrace{\left[\exp(\sqrt{-\lambda}\, L) - \exp(-\sqrt{-\lambda}\, L)\right]}_{\text{can this equal } 0\,?} = 0.$$

Let $z = \sqrt{-\lambda}\, L > 0$

Can $\exp(z) - \exp(-z) = 0$?

No: $\exp(z) > 1 \quad (z > 0)$

$\exp(-z) < 1 \quad (-z < 0)$

Therefore, $c_1 = 0$.

$$\Rightarrow c_2 = -c_1 = 0.$$

So $\varphi(x) = 0$ for every $\lambda < 0$.

$\varphi = 0$ trivial, so there are no negative eigenvalues.

$\underline{\lambda > 0}$: $r_{1,2} = \pm\sqrt{-\lambda} = \pm i\sqrt{|\lambda|}$ $\left(= \pm i\sqrt{\lambda}\right)$

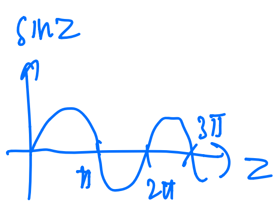
$$\varphi(x) = c_1 \cos\left(\sqrt{|\lambda|}\,x\right) + c_2 \sin\left(\sqrt{|\lambda|}\,x\right)$$

$$\varphi(0) = 0 \longrightarrow 0 = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1$$

$$\varphi(L) = 0 \longrightarrow 0 = c_2 \sin\left(\sqrt{|\lambda|}\,L\right)$$

Trying to avoid $c_2 = 0$, so can $\sin\left(\sqrt{|\lambda|}\,L\right) = 0$?



yes, if $\sqrt{|\lambda|}\,L = n\pi$ ✓, $n = 1, 2, 3, \ldots$

$$|\lambda| = \left(\frac{n\pi}{L}\right)^2$$

$$|\lambda| = \lambda \quad (\lambda > 0)$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

Write $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ since λ depends on our choice of n.

If $\lambda = \lambda_n$, then we can choose $c_2 \neq 0$.

So, if $\lambda = \lambda_n$, then $\varphi(x) = c_2 \sin\left(\sqrt{|\lambda_n|}\, x\right)$

$$= c_2 \sin\left(n\pi x\right), \quad n = 1, 2 \ldots$$

$c_2 \neq 0$

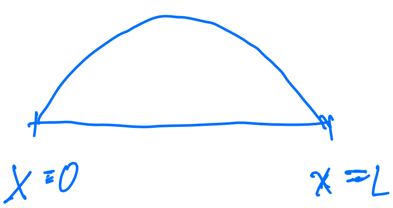
$\sin\left(\sqrt{\lambda_n}\, x\right)$

Given $n = 1, 2, \ldots$, define $\varphi_n(x) = \sin\left(\frac{n\pi}{L} x\right)$

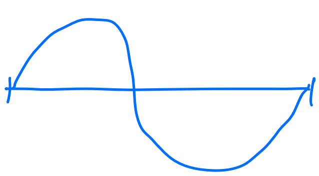
(Choose $c_2 = 1$, arbitrary)

In the end: $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ are eigenvalues

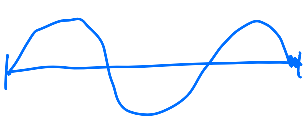
$\varphi(x) = \varphi_n(x) = \sin\left(\frac{n\pi}{L} x\right)$ are associated eigenfunctions.



$n = 1$



$n = 2$



$n = 3$

Summary of step 2 : $\varphi''(x) + \lambda \varphi(x) = 0$

$$\varphi(0) = 0$$

$$\varphi(L) = 0$$

nontrivial solutions φ exist if we choose

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

associated eigenfunctions: $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Observe : $\int_0^L \varphi_n(x)\, \varphi_m(x)\, dx$

$$= \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

HW

$$= \begin{cases} 0, & n \neq m \\ L/2, & n = m \text{ (positive integers)} \end{cases}$$

This relation is called orthogonality.

We say "eigenfunctions are orthogonal".

(In this class, can assume eigenfunctions are orthogonal)

Step 3: General solution and initial conditions

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$

$u(x,t) = \varphi(x)\,T(t) \longrightarrow \varphi''(x) + \lambda\varphi(x) = 0$

$\varphi(0) = 0$

$\varphi(L) = 0$

$T'(t) + \lambda k T(t) = 0$

step 1

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

step 2

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Last step: chose $\lambda = \lambda_n$

$$T'(t) + \lambda_n k T(t) = 0.$$

is a linear, constant-coeff, homogeneous ODE.

Characteristic eqn:

$$T(t) = a \exp(-k \lambda_n t)$$

(a constant)

$$T(t) = a \cdot \exp\left(-k \left(\frac{n\pi}{L}\right)^2 t\right)$$

call this T_n, since it depends on n, and call a as a_n

$$T_n(t) = a_n \exp\left(-k \left(\frac{n\pi}{L}\right)^2 t\right)$$

$$u(x,t) = \varphi(x)\, T(t)$$

choose $\lambda = \lambda_n \Rightarrow$

$$u(x,t) = \phi_n(x)\, T_n(t)$$

$$= a_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \cdot \sin\left(\frac{n\pi x}{L}\right)$$

(this is a sol'n to the PDE!)

(but doesn't satisfy IC's).

Thurs., Feb. 11

- Midterm exam 1 two weeks from today (Feb. 25).
  - practice exam online (website)
  - formula sheet online (website)
  - 100% homework-based.
- HW #3 due Tuesday.
  ((canvas))

Recall: $u_t = k u_{xx}$, $0 < x < L$, $t > 0$

$u(x,0) = f(x)$ (IC's)

$\left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\}$ (BC's)

1.) Separate variables: (transform PDE's to ODE's)

$$T'(t) + k\lambda T(t) = 0$$

$$\varphi''(x) + \lambda \varphi(x) = 0$$

$$\varphi(0) = 0, \quad \varphi(L) = 0$$

$(u(x,t) = T(t)\,\varphi(x))$

2.) Solve eigenvalue problem: find values of λ leading to nontrivial solutions for $\varphi(x)$.

$$\left.\begin{array}{l} \varphi''(x) + \lambda\varphi(x) = 0 \\ \varphi(0) = 0 \\ \varphi(L) = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \\ \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \end{array}$$

eigenfunctions

(λ_n, φ_n) = eigenpair

Notice: eigenfunctions are orthogonal:

$$\int_0^L \varphi_n(x)\,\varphi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \end{cases}$$

3.) Solve the PDE: $\lambda = \lambda_n$

$$T'(t) + \lambda_n k T(t) = 0.$$

$$T_n(t) = \exp(-\lambda_n k t), \quad n = 1, 2, \ldots$$

$$u_n(x,t) = \varphi_n(x) T_n(t) = \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right)$$

is a sol'n to the PDE, and satisfies BC's.

Unless we get lucky, none of these functions satisfy the IC's. $u_n(x,0) = \varphi_n(x)$

Generally: $f(x) \neq \varphi_n(x)$.

Since the PDE is linear, homogeneous, then superposition applies: for <u>any</u> constants $a_1, a_2 \ldots$ then

$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t)$ also satisfies the PDE (and BC's).

this superposition solution is called the <u>general solution</u> (to the PDE).

Though $\{a_k\}_{k=1}^{\infty}$ are arbitrary, we can choose them to satisfy the initial conditions. We do this with orthogonality.

$u(x,0) = f(x)$

$T_n(t) = \exp(-\lambda_n k t)$

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) T_n(0) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

Choose a_n s.t. $f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$

- multiply by $\varphi_{\cancel{k}\,\ell}(x)$ ($\cancel{k}\ \ell$ is some fixed positive integer.)
- integrate from 0 to L..

$$\int_0^L f(x)\,\varphi_\ell(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \varphi_n(x)\,\varphi_\ell(x)\,dx}_{\substack{=0 \text{ if } n\neq \ell \\ = L/2 \text{ if } n=\ell.}}$$

$$\left(= a_1\cdot 0 + a_2\cdot 0 + \cdots a_{\ell-1}\cdot 0 + a_\ell\cdot \tfrac{L}{2} + a_{\ell+1}\cdot 0 + \cdots\right)$$

$$= a_\ell \frac{L}{2}$$

$$\longrightarrow a_\ell = \frac{2}{L}\int_0^L f(x)\,\varphi_\ell(x)\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\,\cancel{\sin}\,\sin\left(\frac{\ell\pi x}{L}\right)dx$$ (everything in this integrand is known!)

These values of a_n ensure that the solution satisfies the IC's.

Solution :

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Separation of variables

Separation of variables has three (major) steps:

1. "Separate variables"
   - Use an educated guess to turn PDEs into ODEs
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2. Satisfy boundary conditions: compute eigenvalues and eigenfunctions
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More examples, I

Example

Solve for $u(x,t)$:

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

1.) Separate variables: ansatz $u(x,t) = T(t)\,\phi(x)$

$$u_t = \phi(x)\,T'(t)$$

$$u_{xx} = \phi''(x)\,T(t)$$

$$u_t = k\,u_{xx} \implies \varphi(x)\,T'(t) = k\,\varphi''(x)\,T(t)$$

$$\frac{T'(t)}{kT(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda \quad \text{(constant)}$$

depends only on t — depends only on x

(intentionally chosen $+\lambda$ instead of $-\lambda$. It doesn't matter, as long as we're consistent.)

ODE's: $T'(t) - \lambda k T(t) = 0$

$\varphi''(x) - \lambda \varphi(x) = 0$

BC's: $u_x(0,t) = 0$

$$u_x(0,t) = \left(\varphi'(x)\,T(t)\right)\Big|_{x=0} = \varphi'(0)\,T(t)$$

$$\varphi'(0)\,\underbrace{T(t)}_{\text{don't want this to be } 0.} = 0$$

$$\implies \varphi'(0) = 0$$

$u_x(L,t)=0$

$\varphi'(L)\,T(t)=0$

$\Longrightarrow \varphi'(L)=0$

$$\begin{cases} T'(t)-\lambda k T(t)=0 \\ \varphi''(x)-\lambda\varphi(x)=0, \quad \varphi'(0)=0, \quad \varphi'(L)=0 \end{cases}$$

2.) Solve an eigenvalue problem (find eigenpairs (λ,φ))

$\varphi''(x)-\lambda\varphi(x)=0, \quad \varphi'(0)=0, \quad \varphi'(L)=0$

Char eqn: $r^2-\lambda=0$

$r=\pm\sqrt{\lambda}$

$\underline{\lambda<0}$: $r=\pm i\sqrt{|\lambda|}$ $\quad (i=\sqrt{-1})$

$\varphi(x)=c_1\cos(x\sqrt{|\lambda|})+c_2\sin(x\sqrt{|\lambda|})$

$\varphi'(x)=-c_1\sqrt{|\lambda|}\sin(x\sqrt{|\lambda|})+c_2\sqrt{|\lambda|}\cos(x\sqrt{|\lambda|})$

$\varphi'(0)=0$

$0=-c_1\cdot 0+c_2\sqrt{|\lambda|}\cdot 1$

$\Longrightarrow c_2=0$

$$\varphi'(L) = 0$$

$$0 = -c_1 \sqrt{|\lambda|}\, \sin\left(L\sqrt{|\lambda|}\right)$$

$$\sin\left(L\sqrt{|\lambda|}\right) = 0$$

$$L\sqrt{|\lambda|} = n\pi \quad , \; n = 1, 2, 3 \ldots$$

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots$$

choose $c_1 = 1$

$$\varphi_n(x) = \cos\left(x\sqrt{|\lambda_n|}\right), \quad n = 1, 2 \ldots$$

$\underline{\lambda = 0}$: $\varphi''(x) = 0 \longrightarrow \varphi(x) = c_1 + c_2 x$

$$\varphi'(x) = c_2$$

$\varphi'(0) = 0 \Rightarrow c_2 = 0$

$\varphi'(L) = 0 \Rightarrow c_2 = 0$

(c_1 arbitrary, choose $c_1 = 1$)

$\varphi(x) = 1$ is an eigenfunction with eigenvalue $\lambda = 0$

$$\left.\begin{aligned} \lambda_0 &= 0 \\ \varphi_0(x) &= 1 \end{aligned}\right\} \text{eigenpair}$$

$\underline{\lambda > 0}$: $\quad r = \pm\sqrt{\lambda}$ (real, distinct)

$$\phi(x) = c_1 \exp(x\sqrt{\lambda}) + c_2 \exp(-x\sqrt{\lambda})$$

$$\phi'(x) = c_1\sqrt{\lambda}\exp(x\sqrt{\lambda}) - c_2\sqrt{\lambda}\exp(-x\sqrt{\lambda})$$

$$\phi'(0) = 0 \Rightarrow c_1\sqrt{\lambda} - c_2\sqrt{\lambda} = 0 \Rightarrow c_1 = c_2$$

$$\phi'(L) = 0 \Rightarrow c_1\sqrt{\lambda}\exp(L\sqrt{\lambda}) - c_1\sqrt{\lambda}\exp(-L\sqrt{\lambda})$$

$$c_1 \underbrace{\sqrt{\lambda}}_{\text{not zero}} \underbrace{\left(z - \frac{1}{z}\right)}_{=0\,?} = 0, \quad z = \exp(L\sqrt{\lambda})$$

$$z - \frac{1}{z} = 0 \Rightarrow z = \pm 1$$

$$\exp(L\sqrt{\lambda}) \neq -1$$

$$\exp(L\sqrt{\lambda}) = 1$$

only if $L\sqrt{\lambda} = 0$

can't happen.

$(L > 0, \sqrt{\lambda} > 0)$

So must choose $c_1 = 0 \rightarrow$ trivial solution no eigenvalues for $\lambda > 0$.

Eigenpairs: $\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2 \ldots$

$\varphi_n(x) = \cos\left(x\sqrt{|\lambda_n|}\right)$

$\lambda_0 = 0$

$\varphi_0(x) = 1$

Observe: orthogonality condition on eigenfunctions:

$$\int_0^L \varphi_n(x)\,\varphi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \geq 1 \\ L, & n = m = 0 \end{cases}$$

formula sheet

Tuesday, Feb. 16

- HW 3 due <u>today</u>
- Quiz 3 due <u>tomorrow</u>
- HW 4 due next Tuesday (Feb. 23)

  (no quiz next week)

Recall:

$$u_t = k u_{xx}, \quad u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,t) = \phi(x)\,T(t) \rightsquigarrow \phi''(x) - \lambda \phi(x) = 0$$

$$T'(t) - k\lambda T(t) = 0$$

(we chose λ for our separation of vars constant instead of $-\lambda$)

Eigenvalue problem: $\phi''(x) - \lambda\phi(x) = 0$

$\phi'(0) = 0, \quad \phi'(L) = 0$

$\lambda = \lambda_0 = 0 \quad \phi_0(x) = 1$

$\lambda = \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, 3, \ldots \quad \phi_n(x) = \cos\left(x\sqrt{|\lambda_n|}\right)$

$= \cos\left(\frac{n\pi x}{L}\right)$

$\lambda > 0$: no eigenvalues.

orthogonality: $\int_0^L \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m = 0 \\ L/2, & n = m \neq 0 \end{cases}$

Final step: satisfy initial conditions.

$$T'(t) - \lambda k T(t) = 0 \xrightarrow{\lambda = \lambda_n} T_n'(t) - \lambda_n k T_n(t) = 0$$

$\Big\downarrow$ char. equ: $r - \lambda_n k = 0$

$$T_n(t) = \exp(\lambda_n k t)$$

$$= \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)$$

$n = 0, 1, 2 \ldots$

For $\lambda = \lambda_n$: $u_n(x,t) = \varphi_n(x) T_n(t)$

$$= \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)$$

is a solution to $u_t = k u_{xx}$, satisfies BC's.

To satisfy IC's: superposition

$$u(x,t) = c_0 \varphi_0(x) T_0(t) + \sum_{n=1}^{\infty} c_n \varphi_n(x) T_n(t),$$

(for arbitrary constants $c_0, c_1, c_2, \ldots$
general solution.

IC's: $u(x,0) = f(x)$

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \underbrace{T_n(0)}_{1 = \exp(0)} = f(x)$$

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) = f(x)$$ (choose c_n to satisfy this)

mult. by $\varphi_m(x)$, integrate.

$$\sum_{n=0}^{\infty} c_n \int_0^L \varphi_n(x)\, \varphi_m(x)\, dx = \int_0^L f(x)\, \varphi_m(x)\, dx$$

$m = 0$: $\quad c_0 \cdot L = \int_0^L f(x)\, \varphi_0(x)\, dx \quad (\varphi_0(x) = 1)$

$$c_0 = \frac{1}{L} \int_0^L f(x)\, \varphi_0(x)\, dx$$

$m > 0$: $\quad C_m \cdot \frac{L}{2} = \int_0^L f(x)\,\varphi_m(x)\,dx$

$$C_m = \frac{2}{L}\int_0^L f(x)\,\varphi_m(x)\,dx$$

<u>Solution</u>: $\quad u(x,t) = \sum_{n=0}^{\infty} c_n\, u_n(x,t)$, where

$$u_n(x,t) = \varphi_n(x)\,T_n(t)$$

$$\varphi_n(x) = \cos\left(x\sqrt{|\lambda_n|}\right)$$

$$T_n(t) = \exp(+\lambda_n k t)$$

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots$$

$$c_0 = \frac{1}{L}\int_0^L f(x)\,\varphi_0(x)\,dx$$

$$c_n = \frac{2}{L}\int_0^L f(x)\,\varphi_n(x)\,dx \quad (n > 0)$$

Alternatively :

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right) \\ c_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ c_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Interpretations :

For the above problem's solution:

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right)$$

weighted sum

spatially oscillatory functions

decaying in time

note: for n large: $\left(\frac{n\pi}{L}\right)^2$ is large

exponentials decay very quickly

for n large: $\cos(n\pi x/L)$ is a very oscillatory function.

Solutions of the heat equation: components of the solution that are highly oscillatory decay very quickly (in comparison to slowly-varying components)

These interpretations are true for the heat equation with any boundary conditions: e.g. for homogeneous Dirichlet conditions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right)$$

More interpretations:

$$u_t = k u_{xx}, \quad u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right)$$

- if $f(x) = \cos\left(\frac{\pi x}{L}\right)$

then $f(x) = \varphi_1(x)$

$$c_0 = \frac{1}{L}\int_0^L f(x)\,\varphi_0(x)\,dx$$

orthogonality

$$= \frac{1}{L}\int_0^L \varphi_1(x)\,\varphi_0(x)\,dx = 0$$

$$c_1 = \frac{2}{L}\int_0^L f(x)\,\varphi_1(x)\,dx$$

orthogonality

$$= \frac{2}{L}\int_0^L \varphi_1(x)\,\varphi_1(x)\,dx = \frac{2}{L}\cdot\frac{L}{2} = 1$$

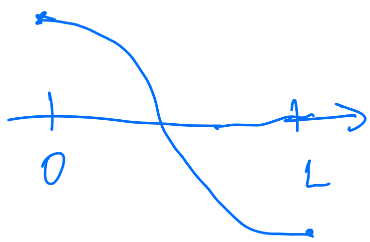
$n \geq 2$: $c_n = \frac{2}{L}\int_0^L f(x)\,\varphi_n(x)\,dx = 0.$

$$\Longrightarrow u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right)$$

$$= 1 \cdot \cos\left(\frac{\pi x}{L}\right) \exp\left(-\left(\frac{\pi}{L}\right)^2 k t\right)$$

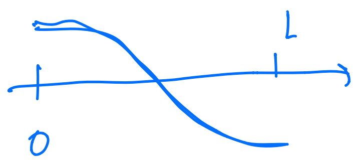
$$= \cos\left(\frac{\pi x}{L}\right) \exp\left(-\left(\frac{\pi}{L}\right)^2 k t\right)$$

$t = 0$:



$f(x) = \cos\left(\frac{\pi x}{L}\right)$
$\shortparallel$
$u(x, 0)$

t small:

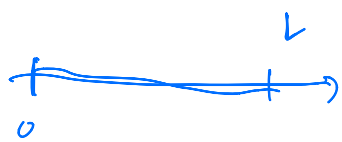


$u(x, t)$

$\exp\left(-\left(\frac{\pi}{L}\right)^2 k t\right) < 1$

but still macroscopic

t large:



$u(x, t)$

$\exp\left(-\left(\frac{\pi}{L}\right)^2 k t\right) \ll 1$

much much smaller than 1.

Compare to equilibrium solutions:

$$\left.\begin{array}{ll} u_t = k u_{xx}, & u(x,0) = f(x) \\ u(0,t) = 0, & u(L,t) = 0 \end{array}\right\} \xrightarrow[\text{solution}]{\text{equilibrium}} \quad \begin{array}{c} \lim_{t\to\infty} u(x,t) \\ \| \\ u_e(x) = 0. \end{array}$$

Recall separation of vars solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

take $t \to \infty$ limit

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \underbrace{\lim_{t\to\infty} \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)}_{0 \text{ for every } n.}$$

$$= 0 = u_e(x)$$

Let's consider another case:

$$u_t = k u_{xx} \qquad u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \; , \quad \frac{\partial u}{\partial x}(L,t) = 0 \Bigg\} \xrightarrow[\text{solution + cons. of energy}]{\text{equilibrium}} u_e(x) = \frac{1}{L}\int_0^L f(x)\,dx$$

Separation of vars:

$$u(x,t) = \sum_{n=0}^{\infty} \cancel{c_x}\, c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)$$

$$\lim_{t\to\infty} u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \underbrace{\lim_{t\to\infty} \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right)}_{\substack{0 \text{ if } n>0 \\ 1 \text{ if } n=0}}$$

$$= c_0 \cos(0) \exp(0)$$

$$= c_0 \underset{\text{sep. of vars}}{=} \frac{1}{L}\int_0^L f(x)\,\phi_0(x)\,dx$$

$$= \frac{1}{L}\int_0^L f(x)\,dx$$

$$= u_e(x)$$

(I.e., equilibrium solutions are consistent with separation of variables)

Thurs, Feb 18

- HW 4 is due Tuesday (Feb 23) midnight MT.
- No quiz next Tues/Wed.
- Tuesday Feb 23: review session
- Next week: office hours Monday 11-12 (usual)
  Wed. 2-3 pm (special)
  no office hours Thurs. Feb. 25
- Midterm exam
  - derive heat equation
  - compute equilibrium solutions
  - solve heat eqn via separation of variables
  - excellent study material: HW.

More examples, II

Example

Solve for $u(x,t)$:

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$

$$u(0,t) = u(L,t), \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t).$$

"Periodic boundary conditions"

Separate variables: $u(x,t) = \varphi(x)\,T(t)$

$$u_t = k u_{xx} \longrightarrow \varphi(x)T'(t) = k\varphi''(x)\,T(t)$$

$$\downarrow$$

$$\frac{T'(t)}{kT(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

for $-\lambda$ an unknown constant.

ODE's: $\varphi''(x) + \lambda\varphi(x) = 0$

$T'(t) + \lambda k T(t) = 0$

BC's: $u(0,t) = u(L,t)$

$\varphi(0)T(t) = \varphi(L)T(t) \longrightarrow T(t)[\varphi(0)-\varphi(L)] = 0$

$\Longrightarrow \varphi(0) = \varphi(L)$

$u_x(0,t) = u_x(L,t)$

$\varphi'(0)T(t) = \varphi'(L)T(t) \longrightarrow T(t)[\varphi'(0)-\varphi'(L)] = 0$

$\Longrightarrow \varphi'(0) = \varphi'(L)$.

$$u(x,t) = T(t)\varphi(x) \Longrightarrow \begin{cases} \varphi''(x) + \lambda\varphi(x) = 0 \\ \varphi(0) = \varphi(L) \\ \varphi'(0) = \varphi'(L) \\ T'(t) + \lambda k T(t) = 0 \end{cases} \quad (\lambda \text{ unknown})$$

Solve eigenvalue problem: Compute λ such that a nontrivial $\varphi(x)$ satisfies:

$\varphi''(x) + \lambda\varphi(x) = 0$

$\varphi(0) = \varphi(L)$

$\varphi'(0) = \varphi'(L)$

char. eqn: $r^2 + \lambda = 0 \longrightarrow r = \pm\sqrt{-\lambda}$

$\underline{\lambda < 0}$: $r = \pm\sqrt{|\lambda|}$, real, distinct.

$$\varphi(x) = c_1 \exp(x\sqrt{|\lambda|}) + c_2 \exp(-x\sqrt{|\lambda|})$$

$$\varphi'(x) = c_1\sqrt{|\lambda|}\,\cancel{x}\,\exp(x\sqrt{|\lambda|}) - c_2\sqrt{|\lambda|}\exp(-x\sqrt{|\lambda|})$$

$$\varphi(0) = \varphi(L) \Longrightarrow c_1 + c_2 = c_1\exp(L\sqrt{|\lambda|}) + c_2\exp(-L\sqrt{|\lambda|})$$

$$\varphi'(0) = \varphi'(L) \Rightarrow c_1\cancel{\sqrt{|\lambda|}} - c_2\cancel{\sqrt{|\lambda|}} = c_1\cancel{\sqrt{|\lambda|}}\exp(L\sqrt{|\lambda|}) - c_2\cancel{\sqrt{|\lambda|}}\exp(-L\sqrt{|\lambda|})$$

call $z = \exp(L\sqrt{|\lambda|})$

$$\Longrightarrow c_1(1-z) + c_2\left(1 - \tfrac{1}{z}\right) = 0$$

$$c_1(1-z) + c_2\left(\tfrac{1}{z} - 1\right) = 0$$

$$\Longrightarrow \begin{pmatrix} 1-z & 1-\frac{1}{z} \\ 1-z & \frac{1}{z}-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is clearly a solution

any nontrivial solutions?

a nontrivial solution exist iff the 2×2 matrix is singular.

$\begin{pmatrix} 1-z & 1-\frac{1}{z} \\ 1-z & -(1-\frac{1}{z}) \end{pmatrix}$ is singular $\Longleftrightarrow \det\begin{pmatrix} 1-z & 1-\frac{1}{z} \\ 1-z & -(1-\frac{1}{z}) \end{pmatrix} = 0.$

$$\det = 0 \implies -(1-z)\left(1-\frac{1}{z}\right) - (1-z)\left(1-\frac{1}{z}\right) = 0$$

$$-2\underbrace{(1-z)\left(1-\frac{1}{z}\right)}_{\text{require } z=1} = 0$$

$$\left.\begin{aligned} z &= \exp(L\sqrt{|\lambda|}) \\ z &= 1 \end{aligned}\right\} \implies L\sqrt{|\lambda|} = 0$$

$L > 0$

$\lambda < 0 \implies \sqrt{|\lambda|} > 0$

So $z = 1$ not possible

$\implies$ matrix is nonsingular

$\implies c_1 = c_2 = 0$ is the only solution

$\implies \phi(x) \equiv 0$ is the only solution

$\implies$ there are no eigenvalues λ satisfying $\lambda < 0$.

$\underline{\lambda = 0}$: $\phi''(x) = 0$

$$r = \pm\sqrt{-\lambda} = 0, 0 \quad \text{(repeated)}$$

$$\phi(x) = c_1 \exp(0x) + c_2 x \exp(0x)$$

$$= c_1 + c_2 x$$

$$\phi'(x) = c_2$$

BC's: $\phi(0)=\phi(L) \Rightarrow c_1 = c_1 + c_2 L \rightarrow c_2 = 0$

$\phi'(0)=\phi'(L) \Rightarrow c_2 = c_2 \rightarrow \checkmark$

c_1 is arbitrary, can choose $c_1 = 1$

$\phi_0(x) = \underline{1}$ is an eigenfunction associated to eigenvalue $\lambda = \lambda_0 = 0$.

$\underline{\lambda < 0}$: $r = \pm\sqrt{-\lambda} = \pm i\sqrt{|\lambda|}$

$$\phi(x) = c_1 \cos\left(x\sqrt{|\lambda|}\right) + c_2 \sin\left(x\sqrt{|\lambda|}\right)$$

BC's: $\phi'(x) = -c_1\sqrt{|\lambda|}\sin\left(x\sqrt{|\lambda|}\right) + c_2\sqrt{|\lambda|}\cos\left(x\sqrt{|\lambda|}\right)$

$\phi(0)=\phi(L) \rightarrow c_1 = c_1\cos\left(L\sqrt{|\lambda|}\right) + c_2\sin\left(L\sqrt{|\lambda|}\right)$

$\phi'(0)=\phi'(L) \rightarrow c_2\cancel{\sqrt{|\lambda|}} = -c_1\cancel{\sqrt{|\lambda|}}\sin\left(L\sqrt{|\lambda|}\right) + c_2\cancel{\sqrt{|\lambda|}}\cos\left(L\sqrt{|\lambda|}\right)$

$$\begin{pmatrix} 1-\cos(L\sqrt{|\lambda|}) & -\sin(L\sqrt{|\lambda|}) \\ \sin(L\sqrt{|\lambda|}) & 1-\cos L\sqrt{|\lambda|} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

nontrivial solutions : require matrix to be singular.

det (matrix) = 0

$$[1-\cos(L\sqrt{|\lambda|})]^2 - - \sin^2(L\sqrt{|\lambda|}) \overset{?}{=} 0$$

$$1-2\cos(L\sqrt{|\lambda|}) + \cos^2 L\sqrt{|\lambda|} + \sin^2 L\sqrt{|\lambda|} = 0$$

$$2 = 2\cos L\sqrt{|\lambda|}$$

$$\cos L\sqrt{|\lambda|} = 1$$

$$L\sqrt{|\lambda|} = 2\pi n, \quad n = 1, 2, \ldots$$

$$\lambda = \lambda_n = \left(\frac{2\pi n}{L}\right)^2$$

what are eigenfunctions?

$$[1-\cos L\sqrt{|\lambda|}]c_1 - (\sin L\sqrt{|\lambda|})c_2 = 0$$

$$\left(\sin L\sqrt{|\lambda|}\right) c_1 + \left[1 - \cos\left(L\sqrt{|\lambda|}\right)\right] c_2 = 0$$

$$\lambda = \lambda_n \Rightarrow \cos L\sqrt{|\lambda_n|} = 1$$

$$\sin\left(L\sqrt{|\lambda_n|}\right) = \sin(2\pi n) = 0.$$

$$\left.\begin{aligned} (1-1)c_1 + 0\cdot c_2 &= 0 \\ 0\cdot c_1 + (1-1)c_2 &= 0 \end{aligned}\right\} \quad 0 = 0 \checkmark$$

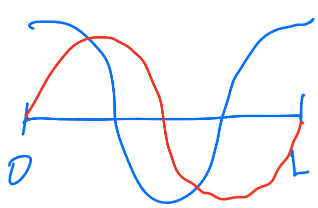
c_1, c_2 arbitrary.

So there are <u>two</u> eigenfunctions corresponding to any eigenvalue.

$$\cos\left(x\sqrt{|\lambda_n|}\right)$$

$$c_1 = 1,\ c_2 = 0 \rightarrow \varphi(x) = \varphi_n(x) = \cos\left(\frac{2\pi n x}{L}\right)$$

$$c_1 = 0,\ c_2 = 1 \rightarrow \varphi(x) = \tilde{\varphi}_n(x) = \sin\left(\frac{2\pi n x}{L}\right)$$



$\varphi_n(x)$ $\qquad (\lambda_n, \varphi_n(x))$: eigenpair

$\tilde{\varphi}_n(x)$ $\qquad (\lambda_n, \tilde{\varphi}_n(x))$: eigenpair.

Eigenfunctions are orthogonal:

$$\lambda_n = \left(\frac{2\pi n}{L}\right)^2, \quad n = 0, 1, 2, 3 \ldots$$

$$\Phi_0(x) = 1$$

$$\Phi_n(x) = \cos\left(\frac{2\pi n x}{L}\right), \quad \tilde{\Phi}_n(x) = \sin\left(\frac{2\pi n x}{L}\right)$$

$$\int_0^L \Phi_n(x)\,\varphi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m = 0 \\ \frac{L}{2}, & n = m > 0 \end{cases}$$

$$\int_0^L \varphi_n(x)\,\tilde{\varphi}_m(x)\,dx = 0 \quad \text{for every } n \geq 0,\ m > 0.$$

$$\int_0^L \tilde{\Phi}_n(x)\,\tilde{\varphi}_m(x)\,dx = \begin{cases} 0, & n \neq m \\ L/2, & m = n \geqslant 0 \end{cases}$$

Superposition: $u(x,t) = u_n(x,t) = T_n(t)\,\varphi_n(x) \quad n \geq 0$

$u(x,t) = \tilde{u}_n(x,t) = T_n(t)\,\tilde{\varphi}_n(x)\,,\ n > 0$

where T_n solves $T_n' + \lambda_n k T_n = 0$

$$T_n(t) = \exp(-\lambda_n k t)$$

General solution: $u(x,t) = \sum_{n=0}^{\infty} a_n \Phi_n(x)\,T_n(t)$

$$+ \sum_{n=1}^{\infty} b_n \tilde{\Phi}_n(x)\,T_n(t)$$

$$= \sum_{n=0}^{\infty} a_n \exp\left(-\left(\frac{2\pi n}{L}\right)^2 k t\right) \cos\left(\frac{2\pi n x}{L}\right)$$

$$+ \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{2\pi n}{L}\right)^2 k t\right) \sin\left(\frac{2\pi n x}{L}\right)$$

..

enforce: $u(\cancel{x},0) = f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x) + \sum_{n=1}^{\infty} b_n \tilde{\varphi}_n(x)$$

orthogonality: $a_0 = \frac{1}{L}\int_0^L f(x)\,\varphi_0(x)\,dx$

$$a_n = \frac{2}{L}\int_0^L f(x)\,\varphi_n(x)\,dx \quad (n>0)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\,\tilde{\varphi}_n(x)\,dx \quad (n>0)$$

Solution: $u(x,t) = \sum_{n=0}^{\infty} a_n \exp\left(-\left(\frac{2\pi n}{L}\right)^2 kt\right)\cos\left(\frac{2\pi n x}{L}\right)$

$$+ \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{2\pi n}{L}\right)^2 kt\right)\sin\left(\frac{2\pi n x}{L}\right)$$

where: $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx \quad (n>0)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n x}{L}\right)dx \quad (n>0)$$