Due today: HW 1 (midnight)

Due tomorrow: Quiz 1 (midnight)

Canvas

Available now: HW 2 (due Feb. 9)

Linearity and superposition

MATH 3150 Lecture 03

February 2, 2021

2.1-2.2

Haberman 5th edition: Sections ~~2.1.2-2~~

PDEs and the heat equation

With $k > 0$ a constant, the heat equation is a PDE

$$u_t = k\, u_{xx}.$$

$u(x,t)$, $k>0$

For a well-posed problem, we also need initial and boundary conditions.

PDEs and the heat equation

L03-S01

With $k > 0$ a constant, the heat equation is a PDE

$$u_t = k\, u_{xx}.$$

For a well-posed problem, we also need initial and boundary conditions.

The PDE above can be equivalently described by *linear operators*.

This alternative description allows us to leverage *linearity* to analyze PDE solutions.

Ex: Let $f(x)$ be a function.

Then $\frac{d}{dx}$ is a linear operation: for any functions f, g, and any constants c_1, c_2, then

$$\frac{d}{dx}\left(c_1 f(x) + c_2 g(x)\right) = c_1 \frac{d}{dx} f(x) + c_2 \frac{d}{dx} g(x)$$

PDEs as linear operators

The heat equation,

$$u_t = k\, u_{xx},$$

can equivalently be written as

$$L\,[u] = 0, \qquad\qquad L[u] := \underbrace{\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)}_{L:\ \text{"take } \partial/\partial t\text{, subtract } k \text{ times } \partial^2/\partial x^2\text{"}} u = u_t - k u_{xx}.$$

L is an "operator"

PDEs as linear operators

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can equivalently be written as

$$L\,[u] = 0, \qquad\qquad L[u] := \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right) u = u_t - k u_{xx}.$$

L above is a (differential) *operator*.

Similar to linear functions, linear operators satisfy linearity conditions.

Definition

An operator L is linear if, given any 2 functions u_1, u_2, and any 2 real scalars c_1, c_2,

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2].$$

A PDE is linear if it can be written as $L[u] = f$ for some linear operator L.

where f is a given, arbitrary function.

"Is an operator L linear?" = Does it satisfy

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2] \ ?$$

Linearity

Example

Show that the heat equation operator, $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$, is linear.

Show $L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]$, where c_1, c_2 are constants, and u_1, u_2 are functions of (x, t).

$$L[c_1u_1 + c_2u_2] = \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(c_1u_1 + c_2u_2)$$

$$= \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2)$$

$$= c_1 \frac{\partial}{\partial t} u_1 + c_2 \frac{\partial}{\partial t} u_2 - k c_1 \frac{\partial^2}{\partial x^2} u_1 - k c_2 \frac{\partial^2}{\partial x^2} u_2$$

$$= c_1 \underbrace{\left(\frac{\partial}{\partial t} u_1 - k \frac{\partial^2}{\partial x^2} u_1 \right)}_{L[u_1]} + c_2 \underbrace{\left(\frac{\partial}{\partial t} u_2 - k \frac{\partial^2}{\partial x^2} u_2 \right)}_{L[u_2]}$$

$= c_1 L[u_1] + c_2 L[u_2]$.

So L is linear.

E.g.. the PDE $u_t = k u_{xx}$ is linear since it's written $L[u] = 0$.

The PDE $u_t = k u_{xx} + x^2 t^2$ is also linear since: $L[u] = f$, where $f = x^2 t^2$.

nonlinear in x, t.
linear/constant wrt u.

Linearity

Example

Show that the heat equation operator, $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$, is linear.

Example

Determine if the operator L defined by $L[u] = \frac{\partial}{\partial x}\left(u^2\right)$ is linear or nonlinear.

Suppose $u = u(x,t)$. $L[u] = \frac{\partial}{\partial x}(u^2) = 2u \cdot u_x$

chain rule

Consider $L[2u] = \frac{\partial}{\partial x}\left((2u)^2\right) = 4\frac{\partial}{\partial x}(u^2)$

$= 4 \cdot L[u] \neq 2L[u]$

Since $L[2u] \neq 2L[u]$, then L is nonlinear.

If $L[u] = \frac{\partial}{\partial x}(2u) \rightarrow$ linear

Homogeneous equations and conditions

In this class, we will exclusively address linear PDEs.

Another useful characterization is whether or not PDEs are homogeneous.

Definition

A linear PDE $L[u] = f$ is homogeneous if $f = 0$.

Ex. $u_t = k u_{xx}$ is a linear, homogeneous PDE.

$u_t = k u_{xx} + x^2 t^2$ is a linear, non-homogeneous PDE.

$u_t = k u_{xx} + Q(x,t)$, $Q \neq 0 \Rightarrow$ non-homogeneous PDE.

$L[u] = Q$

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The *homogeneous* characterization extends to boundary conditions.

Definition

Consider a PDE in one spatial dimension. $u = u(x,t)$

- The initial condition $u(x,0) = f(x)$ is homogeneous if $f = 0$.
- The boundary condition $u(0,t) = T_1(t)$ is a homogeneous Dirichlet condition if $T_1(t) = 0$.
- The boundary condition $u_x(0,t) = T_1(t)$ is a homogeneous Neumann condition if $T_1(t) = 0$.

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Linear, homogeneous equations need not have homogeneous boundary conditions.

Ex. $u_t = k u_{xx}$ (homogeneous PDE)

$u(x,0) = \sin(\pi x)$ (non-homogeneous IC)

$u(0,t) = 0$ (homogeneous Dirichlet BC)

$u_x(L,t) = 3$ (non-homogeneous Neumann BC)

Superposition

The main utility we will get out of these classifications is superposition.

$L[u] = 0$, L linear.

Theorem (Principle of Superposition)

If u_1 and u_2 are both solutions to a linear and homogeneous PDE, then $c_1 u_1 + c_2 u_2$, for any constants c_1 and c_2, is also a solution to the PDE.

The above property applies only addresses the PDE! It does *not* consider initial and/or boundary conditions.

$$L[u_1] = 0 \quad + \quad L[u_2] = 0 \implies L[c_1 u_1 + c_2 u_2] = 0$$

superposition

Examples, I

Example

Verify superposition for the following ODE and solutions u_1, u_2:

$$u''(x) + \frac{\pi^2}{L^2} u = 0, \qquad u(0) = 0, \qquad u(L) = 0.$$

$$u_1(x) = 0, \qquad u_2(x) = \sin\left(\frac{\pi x}{L}\right),$$

Linear, homogeneous ODE: $L[u] = 0$, $L = \left(\frac{d^2}{dx^2} + \frac{\pi^2}{L^2}\right)$

Let c_1, c_2 be constants: $L[c_1 u_1 + c_2 u_2]$

$$\overset{\downarrow}{=} \left(\frac{d^2}{dx^2} + \frac{\pi^2}{L^2}\right)\left(c_1 \cdot 0 + c_2 \cdot \sin\left(\frac{\pi x}{L}\right)\right)$$

$$= c_2 \frac{d^2}{dx^2} \sin\left(\frac{\pi x}{L}\right) + c_2 \frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)$$

$$= -c_2 \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) + c_2 \frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)$$

$= 0 = c_1 L[u_1] + c_2 L[u_2]$ since

u_1 and u_2 are solutions so $L[u_1] = L[u_2] = 0$.

$\Longrightarrow$ $c_1 u_1 + c_2 u_2$ satisfies the ODE. ✓

Examples, I

Example

Verify superposition for the following ODE and solutions u_1, u_2:

$$u''(x) + \frac{\pi^2}{L^2} u = 0, \qquad u(0) = 0, \qquad u(L) = 0.$$

$$u_1(x) = 0, \qquad u_2(x) = \sin\left(\frac{\pi x}{L}\right),$$

Example

Verify superposition for the following PDE and solutions $\cancel{u_1}$ u_3, u_2:

$$u_t = u_{xx}, \qquad u(0,t) = 0, \qquad u(L,t) = 0$$

$$\cancel{u_1(x,t) = 0}, \qquad u_2(x,t) = \exp\left(-(\pi/L)^2 t\right) \sin\left(\frac{\pi x}{L}\right),$$

$$u_3(x,t) = \exp\left(-\left(\frac{2\pi}{L}\right)^2 t\right) \sin\left(\frac{2\pi x}{L}\right).$$

Define $L[u] = \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2}$

Then $L[u_2] = 0$ and $L[u_3] = 0$.

Let $\cancel{c_1}$ c_3, c_2 be constants:

$$L[c_2 u_2 + c_3 u_3]$$

$$\overset{\text{linearity}}{=} c_2 L[u_2] + c_3 L[u_3] = 0.$$

So: $u(x,t) = c_2 \exp\left(-\left(\pi/L\right)^2 t\right) \sin\left(\frac{\pi x}{L}\right)$
$\qquad + c_3 \exp\left(-\left(2\pi/L\right)^2 t\right) \sin\left(\frac{2\pi x}{L}\right)$

satisfies $u_t = u_{xx}$.

Note that u_2, u_3 and $c_2 u_2 + c_3 u_3$ all also satisfy BC's.

(because BC's are homogeneous)

Examples, II

Example

Analyze superposition for the following PDE and solutions u_1, u_2:

wrong as typed

$$u''(x) + \pi^2 u = \cancel{0}\ \pi^2 x, \qquad u(0) = 0, \qquad u(1) = 1$$

$$u_1(x) = x, \qquad u_2(x) = x + \sin(\pi x),$$

Can verify: $L[u] := \frac{d^2}{dx^2}u + \pi^2 u - \pi^2 x$

then $L[u_1] = 0$, $L[u_2] = 0$,

and $u_1(0) = u_2(0) = 0$ (BC)

and $u_1(1) = u_2(1) = 1$ (BC)

But: (i) $L[u]=0$ is not a homogeneous PDE
$(-\pi^2 x)$.

(ii) Right-hand BC ($u(1)=1$) is not homogeneous.

$\Longrightarrow$ $c_1u_1+c_2u_2$ need not satisfy PDE (don't have superposition)

$c_1u_1+c_2u_2$ need not satisfy $u(1)=1$.

E.g. $u_1(1)+u_2(1) = 1+1=2 \neq 1$