Boundary conditions and equilibrium

MATH 3150 Lecture 02

January 28, 2021

1.5: multidimensional version

Haberman 5th edition: Section 1.3-1.5

The heat equation

With $k > 0$ a constant, $u = u(x,t)$, u: Temperature

$$u_t = k\, u_{xx},$$

is the heat equation.

k: given, "thermal diffusivity"

$k > 0$ is important.

The heat equation

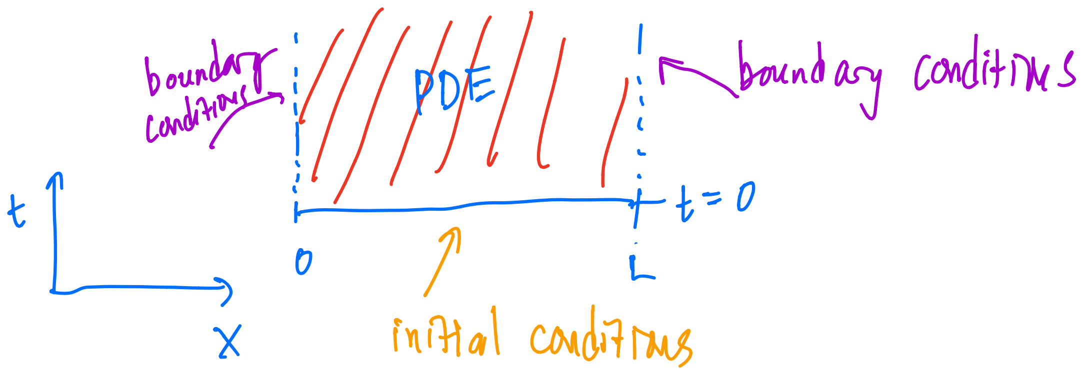
With $k > 0$ a constant,

$$u_t = k\, u_{xx},$$

is the heat equation.

This is not quite enough to fully specify heat diffusion behavior.

We need initial conditions (how does the system start?),
and boundary conditions (what happens at the endpoints of the rod?).



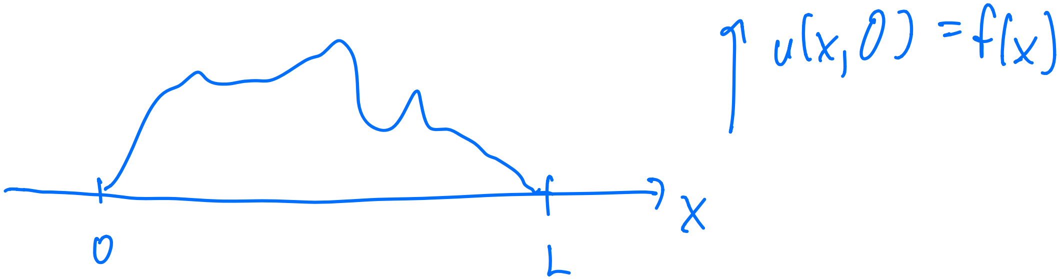
Initial conditions

Initial conditions describes the current, known state of the system.

"current" : "now" : $t=0$

initial conditions amount to a known function $f(x)$

such that $u(x,0)=f(x)$



Boundary conditions ("Dirichlet")

Boundary conditions specify what happens at endpoints of the rod.

One type of boundary conditions are called *Dirichlet* boundary conditions.

Let's fix temperature at endpoints of rod.

endpoints: $x=0$, $x=L$

$u(0,t) = c_1$ (constant) or $u(0,t) = T_1(t)$ (given function)

$u(L,t) = c_2$ (constant) or $u(L,t) = T_2(t)$ (given function)

Need 1 condition @ both $x=0$ and $x=L$.

These kinds of conditions fix the value of the unknown function (u) at boundaries. These kinds of boundary conditions are called Dirichlet cond.

Boundary conditions ("Neumann" and "Robin")

What are alternatives to fixing function values @ boundaries?

"Neumann": fix x-derivative of u at $x=0$, $x=L$.

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{\partial u}{\partial x}(0,t) = u_x(0,t) = T_1(t)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = \frac{\partial u}{\partial x}(L,t) = u_x(L,t) = T_2(t).$$

for the heat equation: recall Fourier's Law $\phi(x,t) = -K_0 \frac{\partial u}{\partial x}$

heat flux.

$\Rightarrow$ fixing $\frac{\partial u}{\partial x}$ is equivalent to fixing heat flux into/out of rod @ boundary.

In particular, if $T_1(t) = 0 \Longrightarrow \frac{\partial u}{\partial x}(0,t) = 0$

$\Longrightarrow$ heat flux @ $x=0$ is 0

$\Longrightarrow$ $x=0$ is insulated.

"Robin": "mixture" of Dirichlet and Neumann.

at $x=0$: $\alpha u(0,t) + \beta \frac{\partial u}{\partial x}(0,t) = T_1(t)$

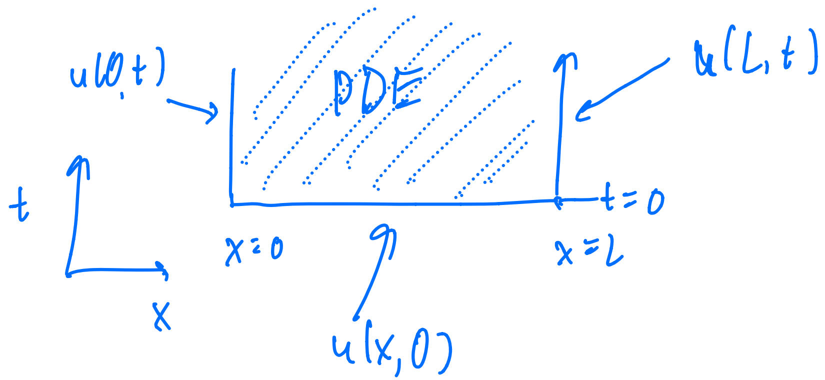
where $\alpha, \beta, T_1(t)$ are given.

It's possible to have, e.g., Dirichlet conditions @ $x=L$ and Neumann conditions @ $x=0$.

The heat equation

A full mathematical model for heat diffusion requires

- the PDE (what happens inside the rod)
- initial conditions (how the rod started)
- boundary conditions (what happens at ends of the rod)



The heat equation

A full mathematical model for heat diffusion requires

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For example, one complete specification, with Dirichlet boundary conditions is

$$\begin{aligned} u_t &= k\, u_{xx}, && x \in (0, L), \quad t > 0 \\ u(x,0) &= f(x), && x \in [0, L] \\ u(0,t) &= T_1(t), && t > 0 \\ u(L,t) &= T_2(t), && t > 0, \end{aligned}$$

where $f(x)$, $T_1(t)$, $T_2(t)$, k, and L are all given.

To "solve" the PDE: compute an (explicit) formula for $u(x,t)$.

Equilibrium

things don't change in time anymore.

Most models: diffusion of heat reaches steady-state equilibrium

Steady-state is reached when time t is large enough.

The solution $u(x,t)$ in this steady state regime is the *equilibrium solution*.

(solving for $u(x,t)$ for every x, t is much harder, we'll do this later)

In steady-state/equilibrium regime $u(x,t)$ doesn't depend on time.

Formally, we're looking for $\lim_{t\to\infty} u(x,t) = u_e(x)$.

equilibrium solution.

To solve for equilibrium state:

1.) Set all time derivatives to 0, $\frac{\partial u}{\partial t} = 0$

2.) Solve the resulting ordinary differential equation for $u_e(x)$.

Equilibrium examples (1/2)

Example

Compute the equilibrium solution for the PDE

$$u_t = k\, u_{xx}, \quad (k>0) \qquad u(x,0) = f(x),$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2.$$

$\frac{\partial u}{\partial t} = 0$

For equilibrium solution: set $\frac{\partial u}{\partial t} = 0$, $\underset{u(x,t)}{u} \leftarrow \underset{u_e(x)}{u_e}$

$0 = k\, u_e''(x)$ (PDE $\to$ ODE)

BC's: $u_e(0) = T_1$, $u_e(L) = T_2$

IC's: don't play a role (conditions @ $t=0$)

Solve ODE: $k\, u_e''(x) = 0$

$u_e(0) = T_1, \quad u_e(L) = T_2.$

$k > 0 \longrightarrow u_e'' = 0$ $\quad u_e' = c_2 \xrightarrow{\text{int.}} u_e = c_2 x + c_1$

$u_e(x) = c_1 + c_2 x.$

use boundary conditions to determine c_1, c_2:

$u_e(0) = T_1 \Longrightarrow c_1 + 0 \cdot c_2 = T_1$

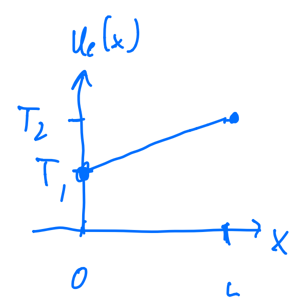
$u_e(L) = T_2 \Longrightarrow c_1 + L \cdot c_2 = T_2$

$\Downarrow$

$c_1 = T_1$

$c_2 = \dfrac{T_2 - T_1}{L}$

$$\boxed{u_e(x) = T_1 + x\left(\frac{T_2 - T_1}{L}\right)}$$



Equilibrium examples (1/2)

Example

Compute the equilibrium solution for the PDE

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2.$$

Example

Compute the equilibrium solution for the PDE

$$u_t = k\,u_{xx}, \quad k>0 \qquad u(x,0) = f(x),$$
$$u_x(0,t) = 0, \qquad u_x(L,t) = 0.$$

1.) Set $\frac{\partial u}{\partial t} \to 0$, $u(x,t) \leftarrow u_e(x)$

PDE: $0 = k u_e''(x)$
BC's: $u_e'(0) = 0$, $u_e'(L) = 0$ $\Big\}$ ODE
IC's: don't play a role (it seems...)

$k u_e''(x) = 0 \xrightarrow{\text{solve}} u_e(x) = c_1 + c_2 x$

$u_e'(x)$

BC's: $u_e'(0) = 0 \Longrightarrow 0 = c_2$
$u_e'(L) = 0 \Longrightarrow 0 = c_2$

$u_e(x) = c_1$ c_1 arbitrary (???)

Appeal to conservation of energy to compute c_1.

$\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0 \Longrightarrow$ rod is insulated from environment.

$\Rightarrow$ energy @ $t=0$ equal energy @ $t=\infty$.

recall: energy density $e(x,t) = c \cdot \rho \cdot u(x,t)$.
c, ρ are constant.

energy @ $t=0$: $\int_0^L e(x,0)\,dx = c\rho \int_0^L u(x,0)\,dx$
$= c\rho \int_0^L f(x)\,dx$ (IC!)

energy @ $t=\infty$: $\int_0^L e(x,\infty)\,dx = c\rho \int_0^L u_e(x)\,dx$
$= c\rho \int_0^L c_1\,dx$

$$= c\rho c_1 L$$

I.e.: $\underbrace{c\rho \int_0^L f(x)\,dx}_{t=0} = \underbrace{c\rho c_1 L}_{t=\infty}$

$$c_1 = \frac{1}{L}\int_0^L f(x)\,dx$$

= "average value of temp. at ~~t~~ 0"

$$\boxed{u_e(x) = \frac{1}{L}\int_0^L f(s)\,ds}$$

Equilibrium examples (2/2)

Example

For what values of (α, β) does

$$u_t = k\, u_{xx}, \qquad u(x,0) = f(x),$$
$$u_x(0,t) = \alpha, \qquad u_x(L,t) = \beta.$$

have a well-defined equilibrium solution?

$\frac{\partial u}{\partial t} \to 0, \quad u \leftarrow u_e(x)$

$$\left.\begin{array}{l} k u_e'' = 0 \\ u_e'(0) = \alpha \\ u_e'(L) = \beta \end{array}\right\}$$

$u_e(x) = c_1 + c_2 x$

$u_e'(0) = \alpha \Rightarrow c_2 = \alpha$

$u_e'(L) = \beta \Rightarrow c_2 = \beta$

$\boxed{\alpha = \beta}$ (for arbitrary α)