Boundary conditions and equilibrium

MATH 3150 Lecture 02

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The heat equation

With $k > 0$ a constant,

$$u_t = k\, u_{xx},$$

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This is not quite enough to fully specify heat diffusion behavior.

We need initial conditions (how does the system start?),
and boundary conditions (what happens at the endpoints of the rod?).

Initial conditions

Initial conditions describes the current, known state of the system.

Boundary conditions ("Dirichlet")

Boundary conditions specify what happens at endpoints of the rod.

One type of boundary conditions are called *Dirichlet* boundary conditions.

Boundary conditions ("Neumann" and "Robin")

The heat equation

A full mathematical model for heat diffusion requires

- the PDE (what happens inside the rod)
- initial conditions (how the rod started)
- boundary conditions (what happens at ends of the rod)

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For example, one complete specification, with Dirichlet boundary conditions is

$$\begin{aligned} u_t &= k\, u_{xx}, & & x \in (0,L), \quad t > 0 \\ u(x,0) &= f(x), & & x \in [0,L] \\ u(0,t) &= T_1(t), & & t > 0 \\ u(L,t) &= T_2(t), & & t > 0, \end{aligned}$$

where $f(x)$, $T_1(t)$, $T_2(t)$, k, and L are all given.

To "solve" the PDE: compute an (explicit) formula for $u(x,t)$.

Equilibrium

Most models: diffusion of heat reaches steady-state equilibrium

Steady-state is reached when time t is large enough.

The solution $u(x,t)$ in this steady state regime is the *equilibrium solution*.

Equilibrium examples (1/2)

Example

Compute the equilibrium solution for the PDE

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2.$$

Equilibrium examples (1/2)

Example

Compute the equilibrium solution for the PDE

$$\begin{aligned} u_t &= k\,u_{xx}, & u(x,0) &= f(x), \\ u(0,t) &= T_1, & u(L,t) &= T_2. \end{aligned}$$

Example

Compute the equilibrium solution for the PDE

$$\begin{aligned} u_t &= k\,u_{xx}, & u(x,0) &= f(x), \\ u_x(0,t) &= 0, & u_x(L,t) &= 0. \end{aligned}$$

Equilibrium examples (2/2)

Example

For what values of (α, β) does

$$\begin{aligned} u_t &= k\, u_{xx}, & u(x,0) &= f(x), \\ u_x(0,t) &= \alpha, & u_x(L,t) &= \beta. \end{aligned}$$

have a well-defined equilibrium solution?