Due Today : HW 0

Due Tomorrow: Quiz 0

on canvas

Due next Tuesday : HW 1 (Feb 2).

The heat equation

MATH 3150 Lecture 01

January 26, 2021

Haberman 5th edition: Section 1.2

Partial Differential Equations (PDEs)

Overall goal of today: derive a PDE governing heat diffusion.

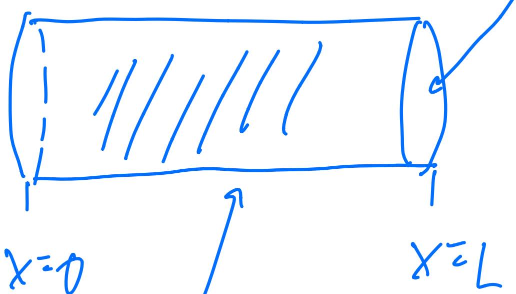
Basic outline:

- Model heat diffusion in an idealized rod
- Energy conservation + Fourier's Law
- Heat equation (PDE) results from taking limits

Conductive rod model

We consider a model for an idealized rod.

surface area A



exterior is perfectly insulated

Goal: model heat/energy flow inside rod.

1.) energy density
$e(x,t)$

time

(doesn't depend on y,z)

units: $\dfrac{\text{energy}}{\text{volume}}$

total energy in rod:

$$E = \int_{\text{rod}} e(x,t)\,dx$$

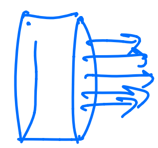
$$= \int_0^L \int_S e(x,t)\,dS\,dx$$

length   surface area

$$= A\int_0^L e(x,t)\,dx$$

(because e is constant across surface area).

2.) heat flux $\varphi(x,t)$. Units: $\dfrac{\text{energy}}{\text{area}\cdot\text{time}}$



3.) "external" sources (external heating/cooling)

$Q(x,t)$: energy per unit volume per unit time due to external sources.

Units: $\dfrac{\text{energy}}{\text{volume}\cdot\text{time}}$

Step 1: Use conservation of energy to relate e, φ, Q.

Conservation of heat energy (differential form)

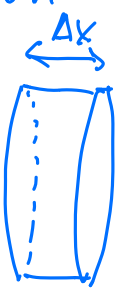
Rate-of-change form:

"e" "φ"

(Rate of energy change) =(energy flux due to flow across boundary)+ (external energy sources)

"Q"

Rate of energy change: consider a small slice of rod



total energy: $A \cdot \int_x^{x+\Delta x} e(x,t)\,dx$

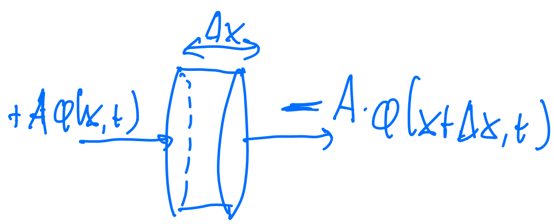
rate of change of energy: $A \cdot \frac{\partial}{\partial t}\left(\int_x^{x+\Delta x} e(x,t)\,dx \right)$

If Δx is very small: then $e(x,t)$ is approximately constant from x to $x+\Delta x$.

I.e.: $\int_x^{x+\Delta x} e(x,t)\,dx \approx \Delta x \cdot e(x,t)$

$\Rightarrow$ total energy rate of change $= A\Delta x \cdot \frac{\partial e}{\partial t}$.

Energy flow due to flux:



rate of energy change

units of φ: $\frac{\text{energy}}{\text{area}\cdot\text{time}}$

rate of energy change due to flux: $-A\varphi(x+\Delta x,t) + A\varphi(x,t)$.

Energy change due to external sources:

$Q(x,t)$: energy imparted to rod per unit volume per unit time.

energy change per unit time: $\int_{\text{vol.}} Q(x,t)\,dx$

$\approx A\cdot\Delta x\cdot Q(x,t)$.

Q is constant over Δx length.

Putting it all together:

$$A\Delta x \cdot \frac{\partial e}{\partial t} = -A\left(\varphi(x+\Delta x, t) - \overbrace{\varphi(x)}^{\varphi(x,t)}\right) + A\cdot\Delta x\, Q(x,t)$$

$$\frac{\partial e}{\partial t} = -\frac{\varphi(x+\Delta x, t) - \overset{\varphi(x,t)}{\cancel{\varphi(x)}}}{\Delta x} + Q(x,t)$$

$$\Delta x \longrightarrow 0$$

$$\underbrace{\frac{\partial e}{\partial t}}_{\frac{\text{energy}}{\text{volume}\cdot\text{time}}} = -\underbrace{\frac{\partial \varphi}{\partial x}}_{\frac{\text{energy}}{\text{area}\cdot\text{time}\cdot\text{length}}} + \underbrace{Q(x,t)}_{\frac{\text{energy}}{\text{volume}\cdot\text{time}}} \longleftarrow$$

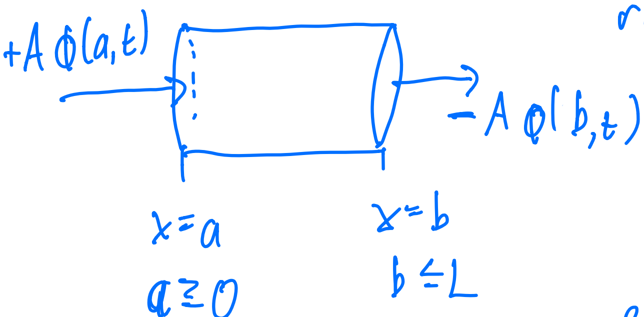
desired relation between Q, e, φ due to energy conservation.

Conservation of heat energy (integral form)

We can obtain the same result using an integrated energy approach.

Rate-of-change form:

$$\left(\text{ Rate of energy change }\right) = \left(\text{ energy flux due to flow across boundary }\right) + \left(\text{ external energy sources }\right)$$



rate of energy change: $\frac{\partial}{\partial t}\left(A\int_a^b e(x,t)\,dx\right)$

e is constant across cross-sectional surface.

energy flux: $-A\left(\phi(b,t)-\phi(a,t)\right)$

$$= -A\left[\int_a^b \frac{\partial \phi}{\partial x}\,dx\right]$$

FTC: $F(b) - F(a) = \int_a^b F'(x)\,dx$

external sources: $\int_{vol} Q(x,t)\,dV$

$$= A\int_a^b Q(x,t)\,dx$$

Q independent of y, z.

conservation of energy: $\frac{\partial}{\partial t}\left(A\int_a^b e(x,t)\,dx\right) = -A\int_a^b \frac{\partial \phi}{\partial x}\,dx + A\int_a^b Q(x,t)\,dx$

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q(x,t)\,dx$$

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t)\right)dx = 0$$

a, b are arbitrary.

I.e, the equation above is true for <u>every</u> a, b.

$\Longrightarrow$ integrand is 0,

$$\Longrightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t)$$

PDE from conservation of energy

We now have a PDE describing conservation of energy:

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t) = 0.$$

(Step 2)

Next goal: rewrite this in terms of a single function. (temperature)

- Relate energy density to temperature via specific heat
- Relate energy flux to temperature via Fourier's Law

Preliminaries: $\rho(x,t)$: mass density of rod $\left(\text{mass}/\text{volume} \right)$

$c(x,t)$: specific heat of rod.

Def'n: amount of energy required to raise 1 unit of mass by 1 temp. unit.

units: $\left(\text{energy}/\text{mass} \cdot \text{temperature} \right)$

Specific heat

Goal: relate energy density e to temperature u.

$e(x,t) = \frac{\text{energy}}{\text{volume}}$ $\quad c(x,t) = \frac{\text{energy}}{\text{mass}\cdot\text{temperature}}$ $\quad \rho(x,t) = \frac{\text{mass}}{\text{volume}}$

$u(x,t)$: temperature.

$$e(x,t) = c(x,t)\cdot\rho(x,t)\cdot u(x,t).$$

treat as known/given.

They are properties of rod.

Fourier's Law

Goal: relate energy flux ϕ to temperature u.

Fourier's Law is based on following observations:

- constant temperature (in space) $\Longrightarrow$ no flux.
- heat flows toward locations of low temperature.
  ("heat flows against temperature gradient")
- rate of heat flow depends on the material.

Quantitative description: $\boxed{\phi(x,t) = -K_0 \frac{\partial u}{\partial x}}$

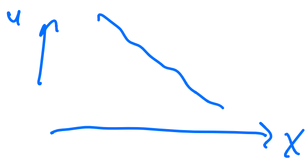
K_0: "diffusion coefficient."

Fourier's Law

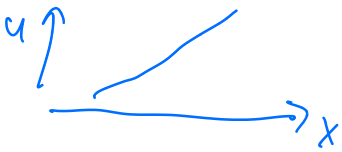
"against"

positive, material-dependent constant.

temp. gradient



$\frac{\partial u}{\partial x} < 0 \Rightarrow \phi > 0 \Rightarrow$ heat flows to right ✓.



$\frac{\partial u}{\partial x} > 0 \Rightarrow \phi < 0 \Rightarrow$ heat flows to left. ✓.

The heat equation

(Energy conservation + specific heat definition + Fourier's Law) yield the heat equation.

$$\frac{\partial e}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q(x,t)$$

$$e(x,t) = c(x,t)\,\rho(x,t)\;u(x,t)$$

$$\varphi = -K_0 \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial t}\Big(c(x,t)\,\rho(x,t)\,u(x,t)\Big) = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q(x,t)$$

$$\frac{\partial}{\partial t}\left(c(x,t)\,\rho(x,t)\,u(x,t)\right) = k_0\, u_{xx} + Q(x,t)$$

The heat equation (in general form)

k_0, c, ρ, and Q are given

u is the unknown. (that we'll solve for)

Common simplifications (that we'll almost always make):

- c and ρ are constant.
- no sources: $Q = 0$

$$\frac{\partial}{\partial t}\left(c \cdot \rho \cdot u(x,t)\right) = k_0\, u_{xx}$$

$$u_t = \frac{k_0}{c\rho}\, u_{xx}$$

define $k := \frac{k_0}{c\rho}$ "thermal diffusivity"

$$\boxed{u_t = k\, u_{xx}, \quad u = u(x,t)}$$

The heat equation.

The heat equation, simplified

The typical simplification of the heat equation we'll use in this class:

$$u_t = k\, u_{xx},$$

with $k > 0$ a constant.

This equation is almost enough to model heat flow

We need initial conditions (what state the rod starts in),

and boundary conditions (what happens at $x=0$, $x=L$?).