The heat equation

MATH 3150 Lecture 01

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Haberman 5th edition: Section 1.2

Partial Differential Equations (PDEs)

Overall goal of today: derive a PDE governing heat diffusion.

Basic outline:

- Model heat diffusion in an idealized rod
- Energy conservation + Fourier's Law
- Heat equation (PDE) results from taking limits

Conductive rod model

We consider a model for an idealized rod.

Conservation of heat energy (differential form)

Rate-of-change form:

$$\text{(Rate of energy change)} = \text{(energy flux due to flow across boundary)} + \text{(external energy sources)}$$

Conservation of heat energy (integral form)

We can obtain the same result using an integrated energy approach.

Rate-of-change form:

$$\begin{aligned}(\text{ Rate of energy change }) =& (\text{ energy flux due to flow across boundary })+ \\ & (\text{ external energy sources })\end{aligned}$$

PDE from conservation of energy

We now have a PDE describing conservation of energy:

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t) = 0.$$

Next goal: rewrite this in terms of a single function.

- Relate energy density to temperature via specific heat
- Relate energy flux to temperature via Fourier's Law

Specific heat

Goal: relate energy density e to temperature u.

Fourier's Law

Goal: relate energy flux ϕ to temperature u.

The heat equation

(Energy conservation + specific heat definition + Fourier's Law) yield the heat equation.

The heat equation, simplified

The typical simplification of the heat equation we'll use in this class:

$$u_t = k\, u_{xx},$$

with $k > 0$ a constant.