Convex sets

Lecture 12

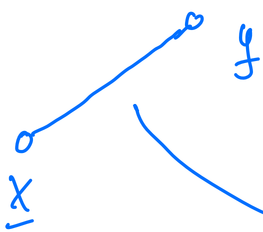
October 26, 2021

Beck, sections 6.1-6.4

Convex sets

<u>Definition</u>: A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall\ \underline{x}, \underline{y} \in C$ and any $\lambda \in (0,1)$, then $\lambda \underline{x} + (1-\lambda)\underline{y} \in C$

Also: C is convex if $\forall\ \underline{x}, \underline{y} \in C$, then $(\underline{x}, \underline{y}) \in C$.

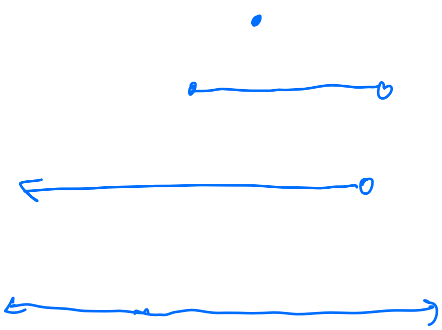


$\{ t\underline{x} + (1-t)\underline{y} : t \in (0,1) \}$

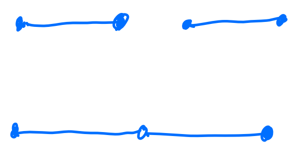
$[\underline{x}, \underline{y}] = (\underline{x}, \underline{y}) \cup \{\underline{x}, \underline{y}\}$

Also: C is convex if, $\forall\, \underline{x}, y \in C$, the line segment connecting $\underline{x}$ and y lies entirely in C.

Examples: in $\mathbb{R}$: convex sets are intervals (possibly infinite)



convex



not convex

$C = \{x\}$ on $\mathbb{R}$

$\forall\, a, b \in C$, $\forall\, \lambda \in (0,1)$, then

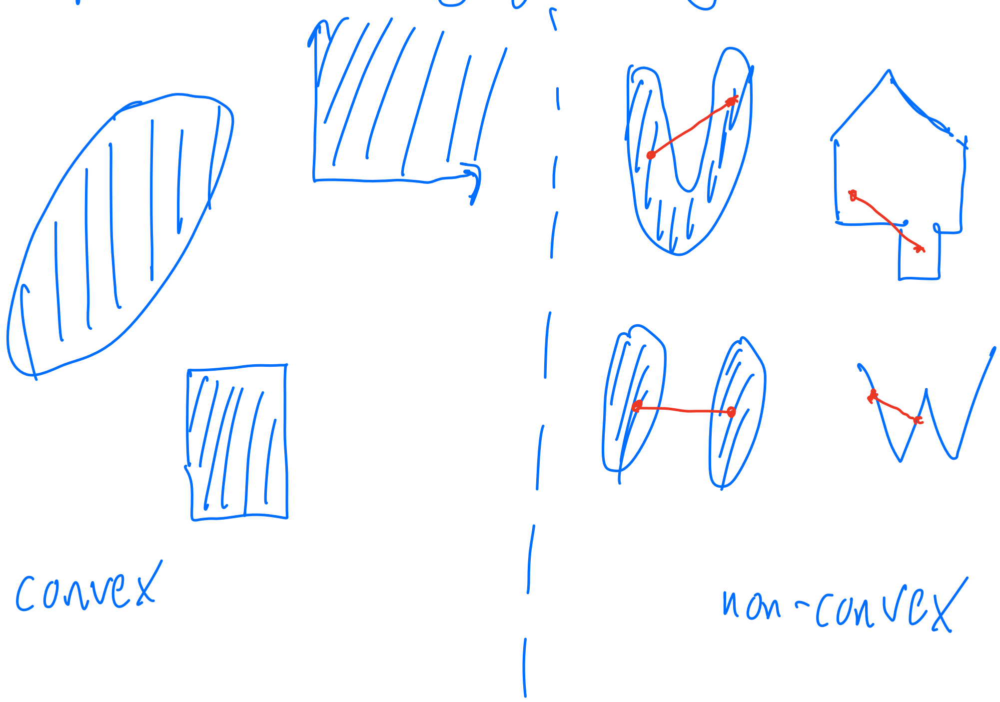
$$\lambda a + (1-\lambda) b \overset{?}{\in} C$$

$a = x$, $b = x$ (no other choices)

$$\lambda a + (1-\lambda) b = \lambda x + (1-\lambda) x = x \in C$$

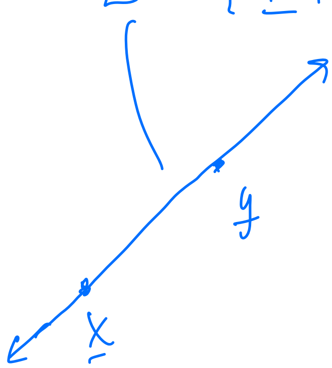
$\Rightarrow$ C is convex.

Examples in $\mathbb{R}^2$ using "geometry".



Example: Show that a line in $\mathbb{R}^n$ is convex.

$L = \{\underline{x} + t\underline{y} \mid t \in \mathbb{R}\}$, (given $\underline{x}, \underline{y}$)



Suppose $\underline{z}_1, \underline{z}_2 \in L$. Let $\lambda \in (0,1)$.

Want to show: $\lambda \underline{z}_1 + (1-\lambda)\underline{z}_2 \in L$

Since $\underline{z}_1, \underline{z}_2 \in L$, then $\exists\, t_1, t_2 \in \mathbb{R}$

s.t. $\underline{z}_1 = \underline{x} + t_1 \underline{y}$, $\underline{z}_2 = \underline{x} + t_2 \underline{y}$.

Then: $\lambda \underline{z}_1 + (1-\lambda)\underline{z}_2 = \lambda(\underline{x} + t_1 \underline{y}) + (1-\lambda)(\underline{x} + t_2 \underline{y}) \overset{?}{\in} L$

$$= \underline{x}\underbrace{[\lambda + (1-\lambda)]}_{1} + \underline{y}\underbrace{[t_1\lambda + (1-\lambda)t_2]}_{t_3}$$

by def. of L

$$= \underline{x} + t_3 \underline{y} \in L$$

$$\Longrightarrow \lambda \underline{z}_1 + (1-\lambda)\underline{z}_2 \in L.$$

Ex. (Hyperplanes)

In $\mathbb{R}^n$, given $\underline{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$, then a hyperplane $H(\underline{a}, b) = H$ is given by

$$H = \{ \underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} = b \}$$

H is convex (for any $\underline{a}, b$)

Proof: Let $\underline{x}, \underline{y} \in H$ (i.e. $\underline{a}^T \underline{x} = b$
$\underline{a}^T \underline{y} = b$)

Let $\lambda \in (0,1)$: $\lambda \underline{x} + (1-\lambda) \underline{y} \overset{?}{\in} H$

i.e, is $\underline{a}^T (\lambda \underline{x} + (1-\lambda) \underline{y}) = b$???

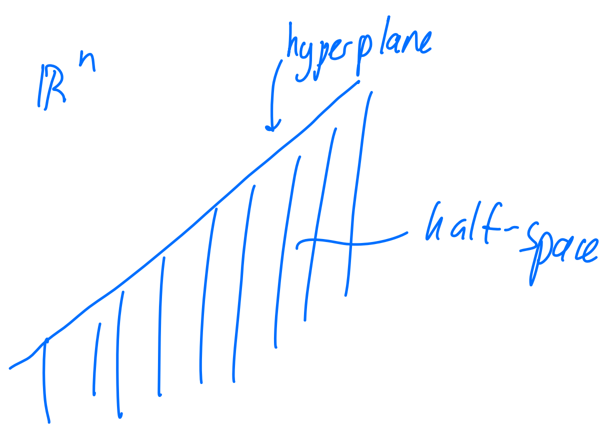
$$\underline{a}^T (\lambda \underline{x} + (1-\lambda)\underline{y}) = \lambda \underline{a}^T \underline{x} + (1-\lambda) \underline{a}^T \underline{y}$$
$$= \lambda b + (1-\lambda) b = b$$

$\Longrightarrow \lambda \underline{x} + (1-\lambda) \underline{y} \in H$.

(so H is convex)

$\underline{\text{Ex}}$ (Half-spaces)

$\mathbb{R}^n$



Given $\underline{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$, then $H^-(\underline{a}, b) = H^-$ is a half-space defined as:

$$H^- = \{ \underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} \leq b \}$$

H^- is convex: Let $\underline{x}, \underline{y} \in H^-$, $\lambda \in (0,1)$.

I.e. $\underline{a}^T \underline{x} \leq b$

$\underline{a}^T \underline{y} \leq b$

if $\lambda = -1$: $\lambda \underline{a}^T \underline{x} = -\underline{a}^T \underline{x} \geq -b = \lambda b$

Then: $\underline{a}^T (\lambda \underline{x} + (1-\lambda) \underline{y}) = \lambda \underline{a}^T \underline{x} + (1-\lambda) \underline{a}^T \underline{y}$

$\leq \lambda b + (1-\lambda) b = b$

$\lambda \in (0,1)$

$\Longrightarrow H^-$ is convex

Ex. (Euclidean balls in $\mathbb{R}^n$ $\cancel{X}$)

$$B[\underline{c}, r] = \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{c}\|_2 \leq r \}$$

$B[\underline{c}, r]$ is convex: Let $\underline{x}, \underline{y} \in B[\underline{c}, r]$, $\lambda \in (0,1)$.

Is $\lambda \underline{x} + (1-\lambda) \underline{y} \in B[\underline{c}, r]$?

$\| [\lambda \underline{x} + (1-\lambda) \underline{y}] - \underline{c} \|_2$

$$= \| [\lambda \underline{x} + (1-\lambda)\underline{y}] - (\lambda \underline{c} + (1-\lambda)\underline{c}) \|_2$$

$$= \| \lambda[\underline{x}-\underline{c}] + (1-\lambda)[\underline{y}-\underline{c}]\|_2$$

$$\leq |\lambda| \, \|\underline{x}-\underline{c}\|_2 + |1-\lambda| \, \|\underline{y}-\underline{c}\|_2$$

triangle inequality

$$= \lambda \|\underline{x}-\underline{c}\|_2 + (1-\lambda)\|\underline{y}-\underline{c}\|_2 \qquad (\lambda \in (0,1))$$

$$\leq \lambda r + (1-\lambda) r = r$$

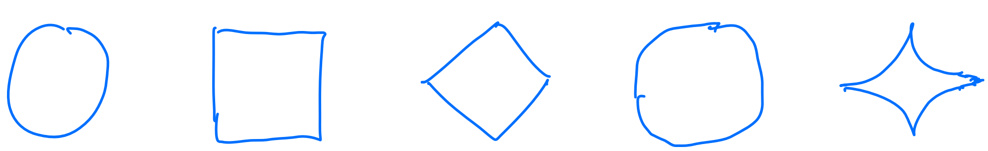
$$\Rightarrow \quad \lambda \underline{x} + (1-\lambda)\underline{y} \in B[\underline{c}, r]$$

Recall: $\underline{x} \in \mathbb{R}^n$, $\quad \|\underline{x}\|_p^p = \sum_{j=1}^{n} |x_j|^p \quad$ (ℓ^p-norm)

Fact: $\|\underline{x}\|_p$ is a (proper) norm iff

$B[\underline{0}, r]$ under the distance $\|\cdot\|_p$ is a convex set.

ℓ^2 $\quad \ell^\infty$ $\quad \ell^1$ $\quad \ell^\pi$ $\quad \ell^{1/2}$



Convex set examples

(Oops)

Convexity-preserving operations

("Properties of convex sets")

<u>Theorem</u>: Intersections of convex sets are convex.

<u>Proof</u> (two sets)

Suppose C_1, C_2 are convex.

$C = C_1 \cap C_2$. Is C convex?

Let $\underline{x}, y \in C$, $\lambda \in (0,1)$

Then $\underline{x}, \underline{y} \in C_1, C_2$

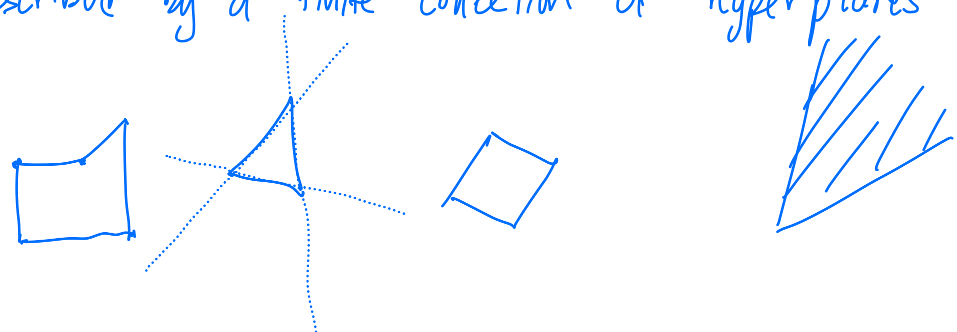
$\Rightarrow \lambda \underline{x} + (1-\lambda)\underline{y} \in C_1 \quad (\underline{x}, \underline{y} \in C_1, \ \lambda \in (0,1))$

$\Rightarrow \lambda \underline{x} + (1-\lambda)\underline{y} \in C_2 \quad (\underline{x}, \underline{y} \in C_2, \ \lambda \in (0,1))$

$\Rightarrow \lambda \underline{x} + (1-\lambda)\underline{y} \in C_1 \cap C_2$. ▨

Unions of convex sets are <u>not</u> (necessarily) convex.

A polytope in $\mathbb{R}^n$ is a set S whose boundary is described by a finite collection of hyperplanes



<u>Ex</u>: A set C is a polytope that is convex ("convex polytope") if it's described, given $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^m$, by

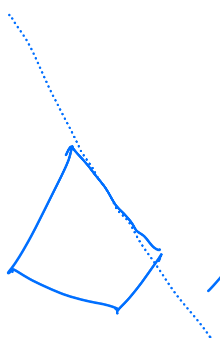
$$C = \{ \underline{x} \in \mathbb{R}^n \mid \underline{A}\underline{x} \leq \underline{b} \},$$

where $\underline{\underline{A}}\,\underline{x} \le \underline{b}$ means that all componentwise inequalities are true.

C is convex because

$$C = \bigcap_{j=1}^{m} \underbrace{\bar{H}(\underline{a}_j, b_j)}_{\substack{\text{half-space} \\ \underline{a}_j^T \underline{x} \le b_j}}, \quad \text{where: } \underline{\underline{A}} = \begin{pmatrix} — \underline{a}_1^T — \\ \vdots \\ — \underline{a}_m^T — \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$



$\underline{a}_3^T \underline{x} = b_3$

More convexity-preserving operations:

Notation: $S, T \subset \mathbb{R}^n$

$$S + T = \{ \underline{x} + \underline{y} \in \mathbb{R}^n \mid \underline{x} \in S, \underline{y} \in T \}$$

"Minkowski"/"set" addition

Proposition: If $C_1, C_2 \ldots C_k$ are convex sets in $\mathbb{R}^n$, and $a_1, a_2 \ldots a_m \in \mathbb{R}$, then

$$\sum_{i=1}^{m} a_i C_i \text{ is convex}$$

$$\left(a \cdot C = \{ a\underline{x} \mid \underline{x} \in C \} \right)$$

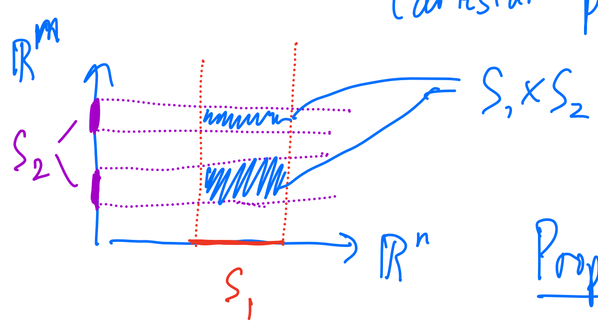
I.e., set addition is convexity-preserving.

Notation: ("Cartesian product")

Given $S_1 \subset \mathbb{R}^n$, $S_2 \subset \mathbb{R}^m$

Then: $S_1 \times S_2 = \{ (\underline{x}, y) \in \mathbb{R}^{m+n} \mid \underline{x} \in S_1, y \in S_2 \}$

Cartesian product



Proposition: The Cartesian product is convexity preserving.
If C_1, C_2 in $\mathbb{R}^n, \mathbb{R}^m$, respectively are convex, then $C_1 \times C_2$ is convex.

Notation: $S \subset \mathbb{R}^n$, $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$

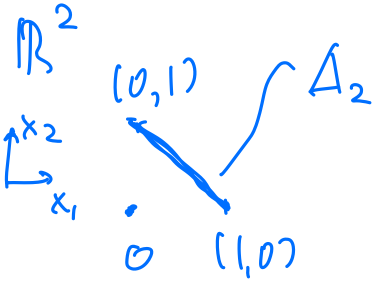
$$\underline{\underline{A}}(S) = \{ \underline{\underline{A}}\,\underline{x} \mid \underline{x} \in S \} \subset \mathbb{R}^m$$

(arbitrary) linear operation on a set S.

Proposition: If $C \subset \mathbb{R}^n$ is convex, and $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$, then $\underline{\underline{A}}(C)$ is convex.

Convex combinatio~~sn~~ns

Recall: in $\mathbb{R}^n$, $\Delta_n = \{ \underline{\lambda} \in \mathbb{R}^n \mid \lambda_i \geq 0,\ \lambda_i \leq 1 \ \forall\, i = 1..n,\ \sum_{i=1}^{n} \lambda_i = 1 \}$



<u>Def</u>: Let $\underline{x}_1 \ldots \underline{x}_k$ be vectors, let $\underline{\lambda} \in \Delta_k$. Then

$$\sum_{i=1}^{k} \lambda_i \underline{x}_i$$ is a <u>convex combination</u> of $\{\underline{x}_i\}_{i=1}^{k}$

(Note: $k=2$ looks familiar... $\underline{\lambda} \in \Delta_2 \Rightarrow \underline{\lambda} = (\lambda_1, 1-\lambda_1)$)

<u>Theorem</u>: Let C be a convex set, and let $\underline{x}_1 \ldots \underline{x}_k \in C$.
Then all convex combinations of $\{\underline{x}_i\}_{i=1}^k$ lie in C:
$\forall\ \underline{\lambda} \in \Delta_k$, then $\sum_{i=1}^k \lambda_i \underline{x}_i \in C$.

<u>Proof</u>: $k=2$: true by definition of convex sets.

Induction: assume it's true for $r < k$.

Goal: prove it's true for $r+1$.

Assume $\underline{\lambda} \in \Delta_{r+1} \Rightarrow \sum_{i=1}^{r+1} \lambda_i \underline{x}_i \overset{?}{\in} C$

$$\sum_{i=1}^{r+1} \lambda_i \underline{x}_i = \lambda_{r+1} \underline{x}_{r+1} + \sum_{i=1}^{r} \lambda_i \underline{x}_i$$

$$= \lambda_{r+1} \underline{x}_{r+1} + (1-\lambda_{r+1}) \underbrace{\sum_{i=1}^{r} \frac{\lambda_i \underline{x}_i}{1-\lambda_{r+1}}}_{\text{"}\underline{v}\text{"}}$$

$$\underline{v} = \sum_{i=1}^{r} \underbrace{\frac{\lambda_i}{1-\lambda_{r+1}}}_{\mu_i} \underline{x}_i =: \sum_{i=1}^{r} \mu_i \underline{x}_i$$

because $\underline{\lambda} \in \Delta_{r+1}$

$$\sum_{i=1}^{r} \mu_i = \sum_{i=1}^{r} \frac{\lambda_i}{1-\lambda_{r+1}} = 1$$

Also: $\mu_i \geq 0 \Rightarrow \underline{\mu} \in \Delta_r$

$\Longrightarrow \underline{v} \in C$ (by inductive hypothesis)

$$\Longrightarrow \underline{x} = \lambda_{r+1} \underbrace{\underline{x}_{r+1}}_{} + (1-\lambda_{r+1}) \underbrace{\underline{v}}_{}$$

elements of C

$\Longrightarrow \underline{x} \in C$ (C is convex). $\blacksquare$

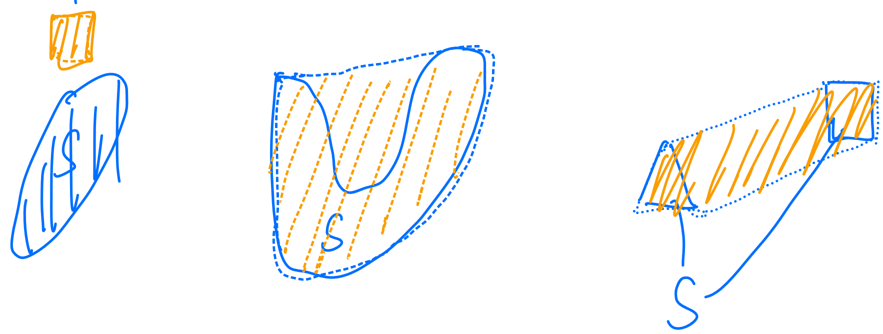
The convex hull

Def: Let $S \subset \mathbb{R}^n$. Then the convex hull of S, $\text{conv}(S)$, is the set of all convex combinations from elements of S.

$$\text{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \underline{x}_i \;\middle|\; \underline{x}_i \in S \;\; \forall\, i = 1..k, \;\; \underline{\lambda} \in \Delta_k, \;\; k \in \mathbb{N} \right\}$$

Geometrically: convex hulls are sets constructed by placing a rubber band around S

Ex: $\boxed{}$: convex hull



Fact: conv(S) is convex.

Proposition: conv(S) is the smallest convex set containing S.

Note: the definition (algebraic) of conv(S) is unwieldly: $\sum_{i=1}^{k} \lambda_i \underline{x}_i$ for k arbitrary.

Turns out: $k = n+1$ is enough ($\underline{x}_i \in \mathbb{R}^n$)

Carathéodory's Theorem

Theorem: Let $S \subset \mathbb{R}^n$. Let $\underline{x} \in \text{conv}(S)$ be arbitrary.

Then: $\exists\ \underline{x}_1, \ldots \underline{x}_{n+1}$, $\underline{\lambda} \in \Delta_{n+1}$ $(\underline{x}_i \in S)$ s.t.

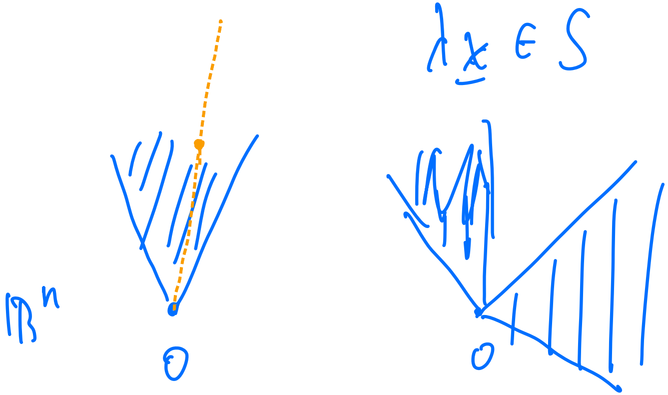
$$\underline{x} = \sum_{i=1}^{n+1} \lambda_i \underline{x}_i$$

Note: Such a representation for $\underline{x}$ is called a <u>barycentric representation</u>.

Convex cones

Def: A set S is a cone if $\forall\ \underline{x} \in S,\ \lambda \geq 0$, then

$\lambda \underline{x} \in S$



(cones need not be convex)

Recall: $\mathbb{R}^k_+ = \{ \underline{x} \in \mathbb{R}^k \mid x_i \geq 0 \quad \forall\ i = 1 \ldots k \}$

Definition: (Conic combination) Let $\underline{x}_1 \ldots \underline{x}_k$ be vectors, and let $\underline{\lambda} \in \mathbb{R}^k_+$. Then $\sum_{i=1}^{k} \lambda_i \underline{x}_i$ is a conic combination.

Proposition: A set S is a convex cone iff all conic combinations lie in S.

Ex: Recall that $\{ \underline{x} \in \mathbb{R}^n \mid \underline{\underline{A}}\,\underline{x} \leq \underline{b} \}$ is a convex polytope.

($\underline{\underline{A}}$, $\underline{b}$ arbitrary)

If $\underline{b} = \underline{0}$, then this set is a convex cone.

Ex: Consider polynomials of degree k on $\mathbb{R}$.

The set of all non-negative polynomials is a convex cone.

Examples

The conic representation theorem